Supremum and Infimum of Set of Real Numbers

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ABSTRACT: In this paper, we study supremum and infimum of set of real numbers.

KEYWORDS: Upper bound, Lower bound, Largest member, Smallest number, Least upper bound, Greatest Lower bound.

I. INTRODUCTION

Let $S$ denote a set (a collection of objects). The notation $x \in S$ means that the object $x$ is in the set $S$, and we write $x \notin S$ to indicate that $x$ is not in $S$. We assume there exists a nonempty set $R$ of objects, called real numbers, which satisfy the ten axioms listed below. The axioms fall in a natural way into three groups which we refer to as the field axioms, the order axioms, and the completeness axiom (also called the least-upper-bound axiom or the axiom of continuity)[10]. Along with the $R$ of real numbers we assume the existence of two operations, called addition and multiplication, such that for every pair of real numbers $x$ and $y$ the sum $x+y$ and the product $xy$ are real numbers uniquely determined by $x$ and $y$ satisfying the following axioms[9]. (In the axioms that appear below, $x$, $y$, $z$ represent arbitrary real numbers unless something is said to the contrary.) In this section, we define axioms[6].

The field axioms

Axiom 1. $x+y = y+x, \ xy = yx$ (commutative laws)
Axiom 2. $x+(y+z) = (x+y) +z, \ x(yz) = (xy)z$ (associative laws)
Axiom 3. $x(y+z) = xy +xz$ (distributive law).
Axiom 4. Given any two real numbers $x$ and $y$, there exists a real number $z$ such that $x+z = y$. This $z$ is denoted by $y-x$; the number $x-x$ is denoted by $0$. (It can be proved that $0$ is independent of $x$.) We write $-x$ for $0-x$ an call $-x$ the negative number of $x$[1].
Axiom 5. There exists at least one real number $x \neq 0$, then there exists a real number $z$ such that $xz = y$. This $z$ is denoted by $y|x$; the number $x|x$ is denoted by $1$ and can be shown to be independent of $x$[2]. We write $x^{-1}$ for $1/x$ if $x \neq 0$ call $x^{-1}$ the reciprocal of $x$.

The order axioms

We also assume the existence of a relation $<$ which establishes an ordering among the real numbers and which satisfies the following axioms[3]:
Axiom 6. Exactly one of the relations $x=y, \ x<y, \ x>y$ holds.
Note. $x > y$ means the same as $y < x$.
Axiom 7. If $x < y$, then for every $z$ we have $x+z < y+z$.
Axiom 8. If $x > 0$ and $y > 0$, then $xy > 0$.
Axiom 9. If $x > y$ and $y > z$, then $x > z$.
Note. A real number $x$ is called positive if $x > 0$, and negative if $x < 0$. We denote by $R^+$ the set of all positive real numbers, and by $R^-$ the set of all negative numbers[4].

Definition 1.1. Let $S$ be a set of real numbers. If there is a real number $b$ such that $x \leq b$ fore every $x$ in $S$, then $b$ is called an upper bound for $S$ and we say that $S$ is bounded above by $b$. 

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We say an upper bound every number greater than b will also be an upper upper bound. If an upper bound b is also a member of S, then b is called the largest member or the maximum element of S. There can be at most one such b. If it exists, we write \( b = \text{max} \, S \).

**Definition 1.2.** Let S be a set of real numbers bounded above. A real number b is called a least upper bound or supremum for S if it has the following two properties:

a) b is an upper bound for S.

b) No number less than b is an upper bound for S.

**The completeness Axiom 10.** Every nonempty set S of real numbers which is bounded above has a supremum; that is, there is a real number b such that b = sup S.

**Comparison property.** Given [6] nonempty subsets S and T of \( \mathbb{R} \) such that \( s \leq t \) for every s in S and t in T. If T has a supremum then S has a supremum and \( \sup \, S \leq \sup \, T \).

### II. MAIN RESULTS

In this section, we prove some theorems on infimum and supremum of set of real numbers.

**Theorem 2.1.** Show that the sup and the inf of a set are uniquely determined whenever they exist.

**Proof.** Given a nonempty set \( S (\subseteq \mathbb{R}) \), and assume sup \( S = a \) and sup \( S = b \), we show \( a = b \) as follows. Suppose that \( a > b \), and thus choose \( \varepsilon = \frac{a - b}{2} \), then there exists a \( x \in S \) such that \( b < \frac{a + b}{2} = a - \varepsilon < x < a \).

which implies that \( b < x \) which contradicts to \( b = \text{sup} \, S \). Similarly for \( a < b \). Hence, \( a = b \).

**Theorem 2.2.** Find the sup and inf of each of the following sets of real number:

(a) All numbers of the form \( 2^{-p} + 3^{-q} + 5^{-r} \), where p, q, and r take on all positive integer values.

**Proof.** Define \( S = \{2^{-p} + 3^{-q} + 5^{-r} : p, q, r \in \mathbb{N}\} \). Then it is clear that \( \sup \, S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \), and \( \inf \, S = 0 \).

(b) \( S = \{ x : 3x^2 - 10x + 3 \leq 0 \} \)

**Proof.** Since \( 3x^2 - 10x + 3 = (x - 3) (3x - 1) \), we know that \( S = \left( \frac{1}{3}, 3 \right) \).

Hence, \( \sup \, S = 3 \) and \( \inf \, S = \frac{1}{3} \).

(c) \( S = \{ x : (x - a) (x - b) (x - c) (x - d) < 0 \} \), where \( a < b < c < d \).

**Proof.** It is clear that \( S = (a, b) \cup (c, d) \). Hence, \( \sup \, S = d \) and \( \inf \, S = a \).

**Theorem 2.3.** Prove the comparison property for supremum.

**Proof.** Since \( s \leq t \) for every \( s \in S \) and \( t \in T \), fixed \( t_0 \in 2 \, T \), then \( s \leq t_0 \) for all \( s \in S \). Hence, by Axiom 10, we know that sup \( S \) exists. In addition, it is clear sup \( S \leq \sup \, T \).

**Remark 2.3.** There is a useful result, we write it as a reference. Let \( S \) and \( T \) be two nonempty subsets of \( \mathbb{R} \). If \( S \subseteq T \) and \( \sup \, T \) exists, then \( \sup \, S \) exists and \( \sup \, S \leq \sup \, T \).

**REFERENCES**

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