Fixed Point of Cyclic Contraction Functions in Banach Spaces

Dr.R.Krishnakumar 1, K.Dinesh 2, T.Amala Kanimozhi 3
Professor and Head, PG and Research Department of Mathematics, Urumu Dhanalakshmi College, Kattur,
Tiruchirappalli, Tamilnadu, India.
Ph.D. Student, PG and Research Department of Mathematics, Urumu Dhanalakshmi College, Kattur,
Tiruchirappalli, Tamilnadu, India.
M.Phil. Student, PG and Research Department of Mathematics, Urumu Dhanalakshmi College, Kattur,
Tiruchirappalli, Tamilnadu, India.

ABSTRACT: In this paper, we proved some fixed point theorems for cyclic mapping with contractive and non-expansive condition on such space analogous to the results presented in the literature.

KEYWORDS: Fixed Point, Contractive Mapping, Cyclic Contraction, Banach spaces.

I. INTRODUCTION

In 2003, Kirk et al. introduced the notation of a cyclic representation and characterized the Banach contraction principle in context of a cyclic mapping. In 2011 Karapinar and Erhan introduced Kannan-type cyclic contraction and Chatterjea-type cyclic contraction. Moreover, they derive some fixed point theorems for such cyclic contractions in complete metric spaces.

Theorem 1.1. (Fixed point theorem for Kannan-type cyclic contraction). Let A and B be nonempty subsets of metric spaces(\(X, d\)) and a cyclic mapping \(T: A \cup B \to A \cup B\) satisfies

\[d(Tx, Ty) \leq k [d(x, Tx) + d(y, Ty)], \forall x \in A, y \in B\]

where \(0 \leq k < 1/2\). Then, \(T\) has a unique fixed point in \(A \cap B\).

Theorem 1.2. (Fixed point theorem for Chatterjea-type cyclic contraction) Let A and B be nonempty subsets of metric spaces(\(X, d\)) and a cyclic mapping \(T: A \cup B \to A \cup B\) satisfies

\[d(Tx, Ty) \leq k [d(x, Tx) + d(y, Ty)], \forall x \in A, y \in B\]

where \(0 \leq k < 1/2\). Then, \(T\) has a unique fixed point in \(A \cap B\).

In this paper we studied some fixed point theorems for cyclic mapping with contractive and non-expansive condition on Banach space.

II. PRELIMINARIES

Definition: 2.1. Let \(K\) be subset of a Banach space \(X\). An operator \(T\) defined on \(K\) is said to belong to class \(D(a, b)\) if

\[\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] \quad (2.1)\]

for all \(x\) and \(y\) in \(K\), where. If an operator \(T\) is in class \(D(k,0)\) with \(0 < k < 1\), then \(T\) is a contraction with \(0 < k < 1\).

Definition: 2.2. Let \(K_1\) and \(K_2\) be closed subsets of a Banach space \(X\). An operator \(T\) defined on \(K\) is said to belong to class \(D(a,b,c)\) if

\[\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|Tx - y\|] \quad (2.2)\]
For all \( x \) and \( y \) in \( K \), where \( 0 \leq a, b, c \leq 1, a + 2b + 2c \leq 1 \) and \( b > 0 \).

**Definition 2.3.** Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). an operator

\[
T : K_1 \cup K_2 \to K_1 \cup K_2 \text{ with } T(K_1) \subset K_2 \text{ and } T(K_2) \subset K_1 \text{ is said to belong to class } D(a, b) \text{ if it satisfies }
\]

\[
\|Tx - Ty\| \leq a\|x - y\| + b\|\|x - Tx\| + \|y - Ty\|\|
\]

(2.3)

for all \( x \in K_1 \) and \( y \in K_2 \), where \( 0 \leq a, b \leq 1 \). It is clear that if \( T \) belongs to the class \( D(k, 0) \) with \( 0 < k < 1 \), then \( T \) is a cyclic contraction.

**Example 2.4.** Let \( K_1 = [0, \frac{1}{2}] \) and \( K_2 = [\frac{1}{3}, 1] \). Define the operator \( T \) as follows:

\[
T(x) = \begin{cases} 
\frac{2}{5}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\
\frac{2}{3}(1 - x), & \text{if } \frac{1}{2} < x \leq 1;
\end{cases}
\]

We will show that \( T \) is in the class of \( D\left(\frac{1}{4}, \frac{1}{4}\right) \). Take \( x \in \left[0, \frac{1}{2}\right] \) and \( y \in \left[\frac{1}{3}, 1\right] \).

Then

\[
\|Tx - Ty\| = \left\| \frac{2}{5} - \frac{2}{3} \right\| = \frac{2}{15} = 0.
\]

Now \( x \in \left[0, \frac{1}{2}\right] \) and \( y \in \left[\frac{1}{2}, 1\right] \). Then

\[
\|Tx - Ty\| = \left\| \frac{2}{5} - \frac{2}{3} \left( y - \frac{2}{3} \right) \right\| = \frac{2}{15} y - \frac{4}{15}
\]

\[
= \frac{1}{4} \left( |x - y| + |\frac{2}{5} + \frac{5}{3} y - \frac{2}{3}| \right)
\]

\[
\leq \frac{1}{4} |x - y| + \frac{1}{4} |x - Ty|
\]

Observe that \( T \) has a unique fixed point \( p = \frac{2}{5} \).

**Definition 2.5.** Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). an operator

\[
T : K_1 \cup K_2 \to K_1 \cup K_2 \text{ with } T(K_1) \subset K_2 \text{ and } T(K_2) \subset K_1 \text{ is said to belong to class } D(a, b, c) \text{ if it satisfies }
\]

\[
\|Tx - Ty\| \leq a\|x - y\| + b\|\|x - Tx\| + \|y - Ty\|\| + c\|\|x - Ty\| + \|Tx - y\|\|
\]

(2.4)

for all \( x \in K_1 \) and \( y \in K_2 \), where \( 0 \leq a, b, c \leq 1 \). It is clear that if \( T \) belongs to the class \( D(k, 0, 0) \) with \( 0 < k < 1 \), then \( T \) is a cyclic contraction.

**III. Main Results**

**Proposition 3.1.** Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). an operator \( T : K_1 \cup K_2 \to K_1 \cup K_2 \) with \( T(K_1) \subset K_2 \) and \( T(K_2) \subset K_1 \) and satisfies (2.4) with \( 0 \leq a, b, c < 1 \) and \( a + 2b + 2c < 1 \). If

\[
F(T) = \{ x \in K_1 \cup K_2 : Tx = x \} \neq \emptyset,
\]

then \( F(T) \) consists of a single point.

**Proof:**

Assume the contrary, that is, let \( z, w \in K_1 \cup K_2 \) be two distinct fixed points of \( T \). Then

\[
\|z - w\| = \|Tz - Tw\| \leq a\|z - w\| + b\|\|z - Tz\| + \|w - Tw\|\| + c\|\|z - Ty\| + \|Tz - w\|\|
\]

which implies \( z = w \), since \( a + 2c < 1 \).

**Theorem 3.2.** Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). suppose that the operator

\[
T : K_1 \cup K_2 \to K_1 \cup K_2 \text{ with } T(K_1) \subset K_2 \text{ and } T(K_2) \subset K_1 \text{ satisfies (2.4) with } 0 \leq a, b, c < 1 \text{ and }
\]

\( a + 2b + 2c < 1 \). Then the sequence \( \{x_n\} \) in \( K_1 \cup K_2 \) satisfies

\[
\lim_{n \to \infty} (x_n - Tx_n) = 0
\]

and the sequence \( \{x_n\} \) converges to the unique fixed point of \( T \).
Proof. Let \( x_0 \in K_1 \). Define \( x_n = T x_{n-1} = T^n x_0 \). Then
\[
\| T x_n - T x_m \| \leq a \| x_n - x_m \| + b \| x_m - T x_m \| + c \| x_n - T x_m \|.
\]
By the triangle inequality we have
\[
\| T x_n - T x_m \| \leq a \{ \| x_n - T x_n \| + \| T x_n - T x_m \| + \| x_m - T x_m \| \} + b \{ \| x_n - T x_n \| + \| x_m - T x_m \| \} + c \{ \| x_n - T x_n \| + \| x_m - T x_m \| + 2 \| T x_n - T x_m \| \}
\]
This implies
\[
\| T x_n - T x_m \| \leq \frac{a + b + c}{1 - a - 2c} \{ \| x_n - T x_n \| + \| x_m - T x_m \| \}.
\]
Observe that from the hypothesis, the right hand side of the inequality tends to 0, as \( n \to \infty \), hence \( \{ T x_n \} \) is a Cauchy sequence. Since \( K_1 \cup K_2 \) is complete, then it converges to a limit, say \( z \in K_1 \cup K_2 \), that is
\[
\lim_{n \to \infty} T x_n = z.
\]
Note that the subsequence \( \{ x_{2n} \} \in K_1 \) and the subsequence \( \{ x_{2n+1} \} \in K_2 \); thus \( z \in K_1 \cap K_2 \neq \emptyset \). Then we employ the triangle inequality and the fact that \( b < 1 \) to get
\[
\| z - T z \| \leq \| T z - T x_n \| \leq a \| z - x_n \| + b \{ \| x_n - T x_n \| + \| T x_n - z \| \} + c \{ \| x_n - T x_n \| + \| z - T z \| + \| x_n - T x_n \| + \| z - x_n \| \}
\]
\[
\leq \frac{a + 2c}{1 - b - c} \| z - x_n \| + \left( \frac{b + c}{1 - b - c} \right) \| x_n - T x_n \|.
\]
It follows from \( \lim_{n \to \infty} x_n = z \) and \( \lim_{n \to \infty} (T x_n - x_n) = 0 \) that \( z \) is the fixed point of \( T \) which is unique by the proposition 3.1.

Corollary 3.3. Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). Suppose that the operator \( T : K_1 \cup K_2 \to K_1 \cup K_2 \) with \( T(K_1) \subset K_2 \) and \( T(K_2) \subset K_1 \) satisfies (2.3) with \( 0 \leq a, b < 1 \) and \( a + 2b < 1 \). Then the sequence \( \{ x_n \} \) in \( K_1 \cup K_2 \) satisfies \( \lim_{n \to \infty} (x_n - T x_n) = 0 \) and the sequence \( \{ x_n \} \) converges to the unique fixed point of \( T \).

Proof: The proof of corollary follows immediately, by taking \( c = 0 \) in the above theorem.

Theorem 3.4. Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). Suppose that the operator \( T : K_1 \cup K_2 \to K_1 \cup K_2 \) with \( T(K_1) \subset K_2 \) and \( T(K_2) \subset K_1 \) satisfies (2.4) with \( 0 \leq a, b, c < 1 \) and \( a + 2b + 2c < 1 \). Then
\[
\text{(i)} \quad T \text{ has a unique fixed point } p \text{ in } K_1 \cap K_2.
\]
\[
\text{(ii)} \quad \| T x - p \| < \| x - p \| \text{ for all } x \in K_1 \cup K_2 \text{ where } p \text{ is the fixed point of } T.
\]

Proof:
\[
\text{(i)} \quad \text{Take a point } x_0 \in K_1. \text{ Define } x_{n+1} = T x_n \text{ for } n = 0, 1, 2, \ldots \text{ then we have}
\]
\[
\| x_{n+1} - T x_{n+1} \| = \| T x_n - T x_{n+1} \| \leq a \| x_n - x_{n+1} \| + b \| x_n - T x_n \| + c \| x_{n+1} - T x_{n+1} \|
\]
\[
+c \| x_n - T x_{n+1} \| + \| T x_n - x_{n+1} \|.
\]
This inequality implies
\[ \|x_{n+1} - Tx_{n+1}\| \leq \left(\frac{a+b+c}{1-b-c}\right)\|x_n - Tx_n\| \leq \cdots \leq \left(\frac{a+b+c}{1-b-c}\right)^n \|x_0 - Tx_0\|. \]

Hence, we obtain
\[ \|x_n - Tx_n\| \to 0 \text{ as } n \to \infty , \text{ implying that } \lim_{n \to \infty} x_n = p , \text{ where } p \text{ is the fixed point of } T. \]

Since the subsequence \( \{x_{2n}\} \subset K_1 \), and the subsequence \( \{x_{2n+1}\} \subset K_2 \), then \( p \in K_1 \cap K_2 \). The uniqueness follows from proposition 3.1.

(ii) Let \( p \) be the fixed point of \( T \) and \( x \in K_1 \cup K_2 \). Then using (2.4) and the triangle inequality, we have
\[ \|Tx - p\| \leq \|Tx - Tp\| + \|Tp - p\| \]
\[ \leq a\|x - p\| + b\|x - Tx\| + \|p - Tp\| + c\|x - Tx\| + \|Tx - p\| \]
\[ \leq a\|x - p\| + b\|x - p\| + \|p - Tp\| + c\|x - p\| + \|Tx - p\| \]

This inequality implies
\[ \|Tx - p\| \leq \left(\frac{a+b+c}{1-b-c}\right)\|x - p\| \leq \|x - p\| \]

as \( \left(\frac{a+b+c}{1-b-c}\right) < 1 \), which completes the proof.

**Corollary 3.5.** Let \( K_1 \) and \( K_2 \) be closed subsets of a Banach space \( X \). Suppose that the operator
\[ T : K_1 \cup K_2 \to K_1 \cup K_2 \text{ with } T(K_1) \subset K_2 \text{ and } T(K_2) \subset K_1 \text{ satisfies (2.3) with } 0 \leq a, b < 1 \text{ and } a + 2b < 1. \]

(i) \( T \) has a unique fixed point in \( K_1 \cap K_2 \).

(ii) \( \|Tx - p\| < \|x - p\| \) for all \( x \in K_1 \cup K_2 \) where \( p \) is the fixed point of \( T \).

**Proof:** The proof of corollary follows immediate, by taking \( c = 0 \) in the above theorem.

**IV. Conclusion**

We have proved some fixed point theorems for cyclic mapping with contractive and expansive conditions in Banach spaces. The presented results generalize the results proved in various spaces and extend some results from the literature.

**References**

7) Karapinar E., "Cyclic contractions and fixed point theorems", Published by Faculty of sciences and mathematics, University of Nis Serbia, Vol.4, pp.777-782, 2012.