

Construction of Scaling Functions of Compactly Supported Wavelets Using MATLAB

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ABSTRACT: In this article MATLAB programming code for creating scaling function of compactly supported wavelet for $L^2(\mathbb{R})$ is presented and results are displayed and discussed.

KEYWORDS: Scaling function, wavelet, MATLAB

I. INTRODUCTION

The collection of all square integrable complex valued functions of \mathbb{R} , in which two functions are identified if they are equal almost everywhere (abbreviated, a.e.), is denoted by $L^2(\mathbb{R})$. With the usual addition, the scalar multiplication and the inner product $\langle f, g \rangle$ of f and g in $L^2(\mathbb{R})$ is defined by $\langle f, g \rangle = \int f(x)\overline{g(x)} dx$, $L^2(\mathbb{R})$ becomes a Hilbert space. An orthonormal wavelet on \mathbb{R} is a function ψ in $L^2(\mathbb{R})$ such that the family $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ forms a complete orthonormal system for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(\cdot) \equiv \left\{ (2)^{\frac{j}{2}} \psi(2^j \cdot - k) : j, k \in \mathbb{Z} \right\}.$$

In 1986, Mallat and Meyer [3] introduced the notion of multiresolution analysis which provides one of the methods of constructing orthonormal wavelets is defined as follows:

Definition 1.1. A multiresolution analysis is a collection $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following axioms:

- (i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,
- (ii) $f(\cdot) \in V_j$ if and only if $f(\theta \cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$,
- (iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
- (iv) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$, and
- (v) There exist functions, $\varphi \in V_0$, called the scaling function, such that the collection $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is a complete orthonormal basis for V_0 .

The following is the definition of compactly supported function [2].

Definition 1.2. Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be a function. The support of f denoted by $\text{supp } f$, is a closure of set $\{x \in \mathbb{R} : f(x) \neq 0\}$. We say f has compact support if $\text{supp } f$ is compact set.

In [2], author shows a concept of developing orthonormal wavelets in $l^2(\mathbb{Z}_n)$, $l^2(\mathbb{Z})$, and then $L^2(\mathbb{R})$. This stepwise development of orthonormal wavelets helps to generate algorithm and then computer programming. The techniques available in MATLAB software even makes the process easier.

MATLAB is scientific computing software that supports numerical programming environment and proprietary programming language by providing graphical user interface. It is a product of The Mathworks of Natick, Massachusetts, USA developed by Cleve Moler [7, 8]. MATLAB is widely used in various fields of mathematics like linear algebra, numerical analysis, differential equation, wavelet analysis and image processing [4, 9]. It is also work as a tool for data analysis, development of algorithm and mathematical experiments. It also supports subroutines written in the other programming languages like C or Fortran.



II. METHODOLOGY

2.1. Following are the two important results presented in [2] that play important role in producing orthonormal wavelets in $L^2(\mathbb{R})$.

Theorem 2.1. Suppose $\{V_j\}_{j \in \mathbb{Z}}$ is a multiresolution analysis with scaling function φ and scaling sequence $u = (u(k))_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$. Define $v = (-1)^{k-1} \overline{u(1-k)}$ for all $k \in \mathbb{Z}$ and ψ by $\psi = \sum_{k \in \mathbb{Z}} v(k) \sqrt{2} \psi(2x - k)$. Then $\{\psi_{j,k}; j, k \in \mathbb{Z}\}$ is a wavelet system in $L^2(\mathbb{R})$.

Theorem 2.2 Suppose $u = (u(k))_{k \in \mathbb{Z}}$ is a sequence satisfying

- (i) $\sum_{k \in \mathbb{Z}} |k|^\epsilon |u(k)| < +\infty$ for some $\epsilon > 0$.
- (ii) $\sum_{k \in \mathbb{Z}} u(k) = \sqrt{2}$.
- (iii) $\{R_{2^j} u; j \in \mathbb{Z}\}$ is an orthonormal set in $l^2(\mathbb{Z})$
- (iv) $\inf_{|\xi| \leq \frac{\pi}{2}} |m(\xi)| > 0$ for $m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} u(k) e^{-ik\xi}$

Then $\prod_{j=1}^\infty m_0(\xi/2^j)$ converges uniformly on bounded subset of \mathbb{R} to a function $\hat{\varphi} \in L^2(\mathbb{R})$. Let $\varphi = (\hat{\varphi})$. For each $j \in \mathbb{Z}$, V_j defined by $V_j = \{\sum_{k \in \mathbb{Z}} z(k) \varphi_{j,k}; z = (z(k))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})\}$ is a multiresolution analysis with scaling function φ and scaling sequence u .

Daubechies's wavelet DN, for each $N \in \mathbb{N}$, for $l^2(\mathbb{Z})$ (for which u has N non zero components) are used to construct compactly supported wavelet for $L^2(\mathbb{R})$, known as Daubechies's wavelet DN for $L^2(\mathbb{R})$. Compactly supported wavelets are widely studied by Daubechies[1]. By applying results mentioned in Theorems 2.1 and 2.2 compactly supported Daubechies's wavelet D6 for $L^2(\mathbb{R})$ can be constructed with vectors $u, v \in l^1(\mathbb{Z})$ having six non zero components defined by

$$\begin{aligned} & (u(0), u(1), u(2), u(3), u(4), u(5)) \\ & = \frac{\sqrt{2}}{32} \in (b + c, 2a + 3b + 3c, 6a + 4b + 2c, 6a + 4b - 2c, 2a + 3b - 3c, b - c), \end{aligned}$$

where $a = 1 - \sqrt{10}$, $b = 1 + \sqrt{10}$ and $c = \sqrt{5 + 2\sqrt{10}}$.

Similarly for constructing compactly supported Daubechies's wavelet D4 for $L^2(\mathbb{R})$ with vectors $u, v \in l^1(\mathbb{Z})$ having four non zero components are defined by

$$(u(0), u(1), u(2), u(3)) = \frac{1}{4\sqrt{2}} \in (1 + a, 3 + a, 3 - a, 1 - a), \text{ where } a = \sqrt{3}.$$

The study related to wavelet sets and also to multiwavelet sets has attracted the attention of several workers [5, 6,10,11,12]. Some of them uses MATLAB techniques to generate wavelet sets.

2.2 M-File Programming. In this section we discuss some of the basic element of the MATLAB that are needed to develop wavelet in $L^2(\mathbb{R})$.

In MATLAB, a program is written and saved in M-file. An M-file is a file having .m extension. There are two types of M-files which can be created in MATLAB: scripts and functions.

Creating a Program: Program code is typed by using a text editor provided with MATLAB and also in any word processing software such as Microsoft Word and Notepad wherefrom the program code can be copied in text Editor of the MATLAB. To create a new M-file, type the word edit at the MATLAB command prompt or click on **File->New->M-file**. This opens an editing window, where program code is typed. As we type in our program, MATLAB keeps track of the line numbers in the left column. To edit an existing M-file, type edit followed by the filename at the MATLAB command prompt:

edit exp1.m or click File->Open -> exp1.m

MATLAB opens the file exp1.m in a new window for editing the saved program code.

Saving Programs: To save a program, click **File -> Save in the Editor/Debugger**. A **Save file as** dialog box appears. Enter a filename with .m extension in the File name field and click OK.

However, a function M-file is saved with the same name as that of function name.

To run a Program: To run a program place the respective M-file in the current directory, or make sure that it is on the MATLAB path. The MATLAB path defines those directories that keep all executing M-files. The path includes all the directories that contain functions provided with MATLAB. To run the program, type the name of the M-file without extension at the MATLAB command prompt:

>>exp1



M-File Scripts: A MATLAB script M-file contains a series of MATLAB statements and execute them when the M-file is run at the command prompt. MATLAB scripts do not accept input arguments or return output arguments. They store variables in a workspace that is shared with other scripts and with the MATLAB command line interface.

M-File Functions: MATLAB M-file functions can accept input arguments and return output arguments. MATLAB functions store variables in a workspace internal to the function.

III. EXPERIMENTAL RESULTS

In this section all the MATLAB program coding of script file and functions are presented and subsequently the result that appears on the screen after running these programs as an image shown here.

3.1. MATLAB program code for constructing scaling function of D4 wavelet

%M file funeta_0.m

```
function val=funeta_0(x);
clearval
if((0<=x)&(x<1))
val=1;
else
val=0;
end;
```

%Script file eta.m

```
m=1;
for x=-1:.01:5
eta(1,m)= funeta_0(x);
m=m+1;
end;
q=-1:.01:5;
plot(q,eta);
ylim([-1 2])
```

%M file funeta_2.m

```
function val=funeta_2(x);
a=sqrt(3);
b=1/(4*sqrt(2));
u=b*[1+a^3+a^3-a^1-a];
clearval
clear sum
sum=0;
for k=0:3
sum=sum+u(1,k+1)*funeta_1(2*x-k);
end;
val=sum;
```

```
function val1=funeta_1(x);
a=sqrt(3);
b=1/(4*sqrt(2));
u=b*[1+a^3+a^3-a^1-a];
clear sum
clear val1
sum=0;
for k=0:3
sum=sum+u(1,k+1)*funeta_0(2*x-k);
end;
val1=sum;
```



```
function val=funeta_0(x);
clear val
if((0<=x)&(x<1))
val=1;
else
val=0;
end;
```

%Script file eta1.m

```
m=1;
for x=-1:.001:5
ett(1,m)=funeta_2(x);
m=m+1;
end;
q=-1:.001:5;
plot(q,ett);
ylim([.2 8])
```

%M file funeta_4.m

```
function val4=funeta_4(x);
a=sqrt(3);
b=1/(4*sqrt(2));
u=b*[1+a^3+a^3-a^1-a^1];
clear val4
clear sum
sum=0;
for k=0:3
sum=sum+u(1,k+1)*funeta_3(2*x-k);
end;
val4=sum;
```

```
function val3=funeta_3(x);
a=sqrt(3);
b=1/(4*sqrt(2));
u=b*[1+a^3+a^3-a^1-a^1];
clear val3
clear sum
sum=0;
for k=0:3
sum=sum+u(1,k+1)*funeta_2(2*x-k);
end;
val3=sum;
```

%Script file eta2.m

```
m=1;
for x=-1:.001:5
ett(1,m)=funeta_4(x);
m=m+1;
end;

q=-1:.001:5;
plot(q,ett);
ylim([-1 .4])
```



%M file funeta_6.m

```
function val6=funeta_6(x);
a=sqrt(3);
b=1/(4*sqrt(2));
u=b*[1+a 3+a 3-a 1-a ];
clear val6
clear sum
sum=0;
for k=0:3
sum=sum+u(1,k+1)*funeta_5(2*x-k);
end;
val6=sum;
```

function val5=funeta_5(x);

```
a=sqrt(3);
b=1/(4*sqrt(2));
u=b*[1+a 3+a 3-a 1-a ];
clear val5
clear sum
sum=0;
for k=0:3
sum=sum+u(1,k+1)*funeta_4(2*x-k);
end;
val5=sum;
```

%Script file eta2.m

```
m=1;
for x=-1:.05:5
ett(1,m)=funeta_6(x);
m=m+1;
end;
q=-1:.05:5;
plot(q,ett);
ylim([-1 .2])
```

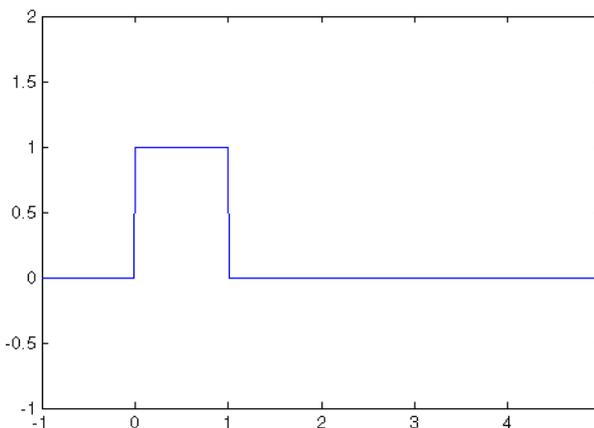


Fig 1: $\eta_0(x)$ for D4

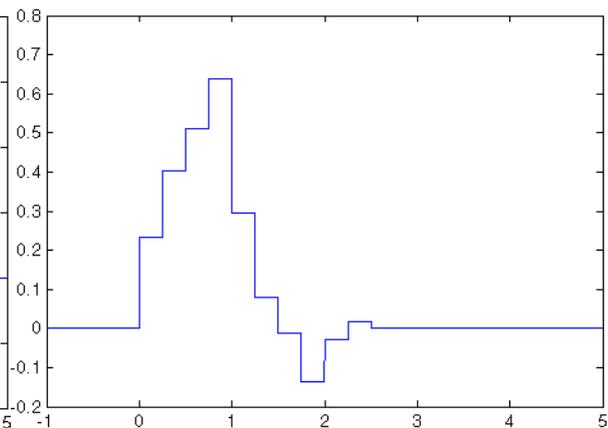


Fig 2: $\eta_2(x)$ for D4

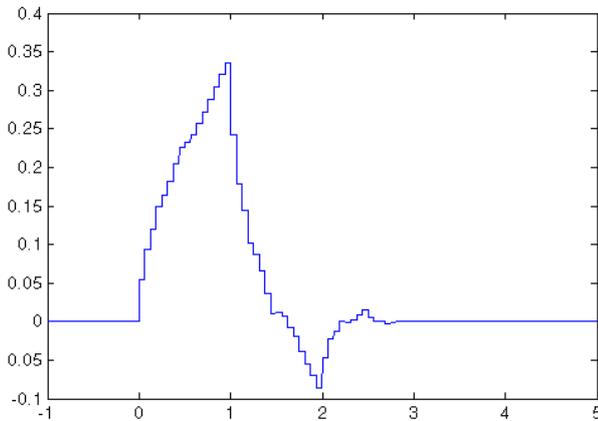


Fig 3: $\eta_4(x)$ for D4

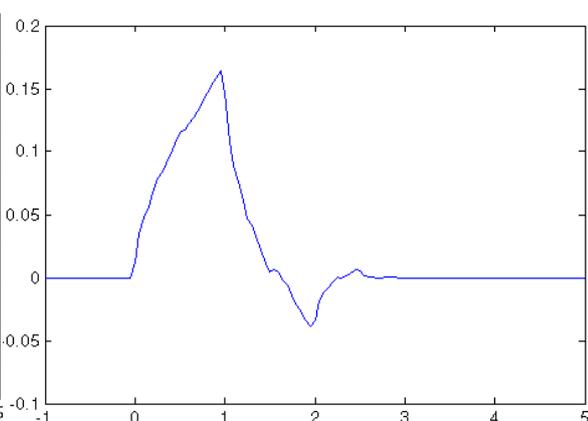


Fig 4: $\eta_6(x)$ for D4 (φ for D4)

3.2. MATLAB program code for constructing scaling function of D6 wavelet

Taking the following values of a, b and c that are used in defining scaling sequence components for D6 wavelet one can easily generate scaling function.

$$a=1-\sqrt{10};$$

$$b=1+\sqrt{10};$$

$$c=\sqrt{5+(2*\sqrt{10})};$$

$$u=(\sqrt{2}/32)*[b+c \ 2*a+3*b+3*c \ 6*a+4*b+2*c \ 6*a+4*b-2*c \ 2*a+3*b-3*c \ b-c];$$

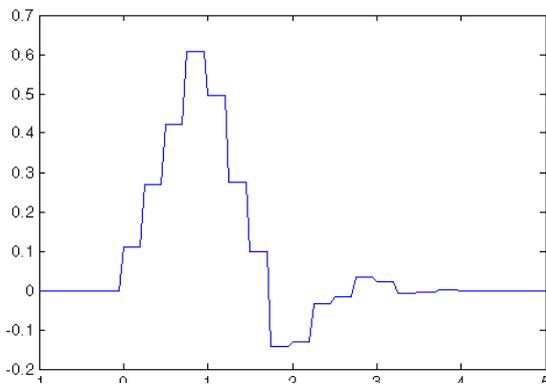


Fig 5: $\eta_2(x)$ for D6

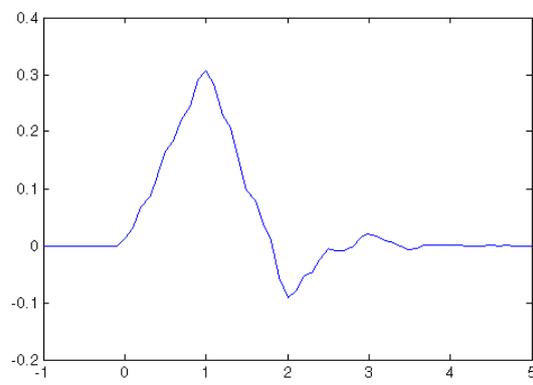


Fig 6: $\eta_4(x)$ for D6

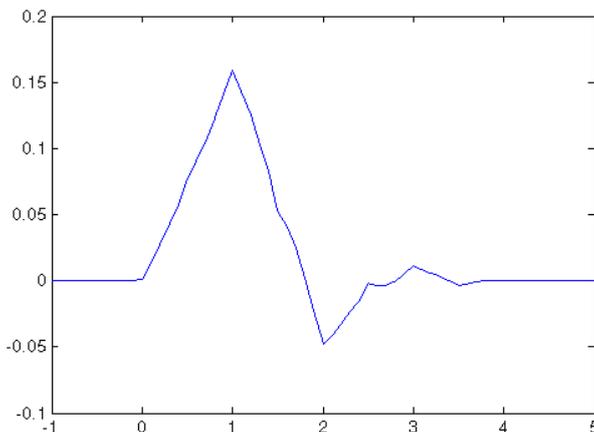


Fig 7: $\eta_6(x)$ for D6 (φ for D6)



It is clear that from smoothness of the scaling function increases by increasing number of iterations for the same wavelet and also by increasing number of components from D4 wavelet to D6 wavelets.

IV. CONCLUSION

In a similar way Daubechies's wavelet $DN, N \in \mathbb{N}$, can be used to construct scaling functions for compactly supported wavelet for $L^2(\mathbb{R})$. With minor changes in the coding given above these wavelets can be easily generated using MATLAB. These programing code further can be extended to construct compactly supported wavelet with the help of scaling function.

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