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DEMONSTRATING THE INVARIANCE PROPERTY OF HOTELLING T^2 STATISTIC

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Abstract: Numerous control chart procedures were developed and their widespread applicability as vital tools in statistical process control cannot be overemphasized. Under normal circumstances, process monitoring and control should include several interrelated process variables. The Hotelling T^2 statistic being one of the most popular control chart procedures can be utilized to detect whether the observed disturbance in a given process is as a result of some perturbations in the mean vector of the quality characteristics or in their variability. One of the most notable features of Hotelling T^2 statistic is its *invariance property*. This article demonstrates this important property using a real dataset obtained from a well-known cement production company in Nigeria.

Keywords: Process Control, Hotelling's T^2 Statistic, Decomposition, Matrix Permutation.

I. INTRODUCTION

Multivariate statistical process control (MSPC) is of immense importance and to larger extent being applied in production companies as well as manufacturing industries where data is collected and monitored on many interrelated process variables. Practically, most of the quality characteristics to be controlled and monitored are dependent. However, ignoring the interrelationship existing between them will be inappropriate. More so, the multivariate control charting opens up opportunity in considering the correlation among process variables. The complexity in the application of MSPC charting technique in terms of understanding the multivariate properties itself proved to be a challenging task to practitioners. The application of MSPC requires an understanding of two distinct phases in control charting procedures namely, phase I and phase II operations. Woodall [8], believes that, it is worthwhile to differentiate between the applications of the two phases. Phase I is aimed at determining whether the process was actually under control given that the historical dataset is properly collected. It is during this phase that the establishment of the initial control limits is done in order to bring the process under control. In addition, if there is no information about the in-control process parameters μ_0 and Σ_0 both parameters could be estimated from historical dataset. Whereas in phase II, the goal is to investigate whether the process remains in-control when a future observation is drawn, in this phase, the control charts are used for monitoring and detecting any changes in the process mean vector or covariance matrix which are the in-control process parameters.

Quite number of researches have discussed in great detail multivariate statistical control procedures; for further details we refer an interested reader to the works of Bersimis *et al*. [1]. The Hotelling T^2 chart, being one of the most important statistical process control tools aimed at monitoring the mean vector of quality characteristics, has been presented in the works of Lowry and Montgomery [2].

In this article our focus is to demonstrate the invariance property of Hotelling's T^2 statistic and the paper is organized as follows. In the next section we present the basics of Hotelling's T^2 control charts, after which we show how the statistic can be decomposed. In the same section II we also demonstrate the invariance property of Hotelling's T^2 statistic. In section III, we provide a numerical example to further boost the reader's understanding of the invariance property. Discussion of results and concluding remarks are presented in section IV.

II. HOTELLING'S T^2 CONTROL CHART

Among the popular multivariate process monitoring and control procedures; the Hotelling's T^2 control chart is in the forefront. In this section, we intend to present the Hotelling's T^2 control chart in relation to the quality characteristics of a certain process. Let us assume there are two quality characteristics (x_1 and x_2) that need to be monitored in a given process and let these characteristics be jointly normally distributed random variables. Let us also assume that, $\boldsymbol{\mu}$ and

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$\boldsymbol{\Sigma}$ respectively represent the mean vector and covariance matrix of the quality characteristics under in-control conditions; which can be computed using:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

Given that a process is presumed to be in-control, the estimates $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ can be obtained from the preliminary samples taken. Hence,

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \bar{\mathbf{x}} \tag{2.1}$$

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_2^2 \end{pmatrix} = \begin{pmatrix} s_1^2 & s_{12} \\ s_{21} & s_2^2 \end{pmatrix} = \mathbf{S} \tag{2.2}$$

According to Montgomery [6], **S** is an unbiased estimator of $\boldsymbol{\Sigma}$ when the process is in-control. Thus, the test statistic for an individual observation becomes;

$$T^2 = \left(\mathbf{x} - \bar{\mathbf{x}}\right)' \boldsymbol{\Sigma}^{-1} \left(\mathbf{x} - \bar{\mathbf{x}}\right) \tag{2.3}$$

When the chart is used for monitoring future production, the control limits to be used is given by:

$$ucl = \frac{p(n+1)(n-1)}{n(n-1)} F_{p,\,n-1}^{\alpha} \tag{2.4}$$

Where, $F_{p,\,n-1}^{\alpha}$ is the value of F-distribution with p and (n-1) degrees of freedom at α significance level.

2.1 Decomposition of T^2 Statistic

Mason *et al*. [4, 5] as well as Runger *et al*. [7] have been unarguably among the famous researches outlining some well informed approaches for the decomposition of Hotelling T^2 statistic. Both researches presented a unique way of decomposing the T^2 statistic; for instance, the procedure for the decomposition of Hotelling T^2 statistic has been done by Mason *et al*. [4, 5] using two different terms, namely, the conditional and unconditional terms as follows.

$$T^2 = T_r^2 + T_{s.r}^2 \tag{2.5}$$

where

$$T_r^2 = \frac{(x_i - \bar{x})^2}{s_r^2} \tag{2.6}$$

and

$$T_{s.r}^2 = T^2(x_r, x_s) - T^2(x_r) \tag{2.7}$$

On the other hand, the procedure employed by Runger *et al*. [7] involved the decomposition of the T^2 statistic into components that reflect the contribution of each individual variable; that is

$$d_r = T^2 - T_{(r)}^2 \tag{2.8}$$

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Where the statistic, d_r is an indicator of the relative contribution of the r^{th} variable to the overall statistics; whereas, $T^2_{(r)}$ is the value of T^2 statistic for all other quality variables except the r^{th} one. Thus,

$$T^2_{(r)} = n\left(x_r - \bar{x}\right)' \Sigma^{-1}\left(x_r - \bar{x}\right) \tag{2.9}$$

2.2 The Invariance Property of Hotelling's T^2 statistic

The invariance principle seeks what always remains the same or invariant. The invariance property of permuting T^2 components guarantees that each ordering of an observation vector will produce the same overall T^2 value, since there are $p! = p(p-1)(p-2)\ldots(2)(1)$ permutations of the components of vectors $\left(x_1, x_2, ----- x_p\right)$ this implies that we can partition a T^2 value in p! different ways. To illustrate, suppose p = 3, then can be decomposed in 6 possible ways as follows:

$$T^2 = \begin{cases} T_1^2 + T_{2.1}^2 + T_{3.12}^2 \\ T_1^2 + T_{3.1}^2 + T_{2.13}^2 \\ T_2^2 + T_{1.2}^2 + T_{3.12}^2 \\ T_2^2 + T_{3.2}^2 + T_{1.23}^2 \\ T_3^2 + T_{1.3}^2 + T_{2.13}^2 \\ T_3^2 + T_{2.3}^2 + T_{1.23}^2 \end{cases}$$

According to Mason and Young [3], T^2 value cannot change by permuting the component of the observation vector, hence:

$$T^2(x_1, x_2, \cdots, x_p) = (x_2, x_1, \cdots, x_p)$$

III. ILLUSTRATIVE EXAMPLE

In this section we present an illustrative example aimed at demonstrating the invariance property of Hotelling's T^2 statistic. The data for this research (see APPENDIX 1) was collected from the production line of West African Portland Cement Company (WAPCO) – a cement production company in Northern Nigeria. Three variables (X_1, X_2, and X_3) representing temperature readings from three boiler machines were monitored and under each twenty five observations were recorded at different time periods. The values of Hotelling's T^2 statistic obtained from the observations on APPENDIX 1 can be found on APPENDIX 2. Thus:

$$\mathbf{S} = \begin{pmatrix} 54 & 0.958 & 20.583 \\ 0.958 & 4.84 & 2.963 \\ 20.583 & 2.963 & 22.993 \end{pmatrix}, \quad \mathbf{X}' = \begin{pmatrix} 507 & 516 & 527 \end{pmatrix} \text{ and } \bar{\mathbf{X}} = \begin{pmatrix} 525 \\ 513.36 \\ 538.92 \end{pmatrix}$$

In order to demonstrate the invariance property of T^2, we consider

$$T^2 = (x - \bar{x})' s^{-1} (x - \bar{x}) = 10.695$$

Now, to see the contribution of each component, the T^2 above is decomposed as follows:

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$$T^2 = \begin{cases} T_1^2 + T_{2.1}^2 + T_{3.12}^2 = 6.000 + 1.586 + 3.109 = 10695 \\ T_1^2 + T_{3.1}^2 + T_{2.13}^2 = 6.000 + 1.668 + 3.027 = 10.695 \\ T_2^2 + T_{1.2}^2 + T_{3.12}^2 = 1.440 + 6.146 + 3.109 = 10.695 \\ T_2^2 + T_{3.2}^2 + T_{1.23}^2 = 1.440 + 8.283 + 0.972 = 10.695 \\ T_3^2 + T_{1.3}^2 + T_{2.13}^2 = 6.179 + 3.544 + 0.972 = 10.695 \\ T_3^2 + T_{2.3}^2 + T_{1.23}^2 = 6.179 + 1.489 + 3.027 = 10.695 \end{cases}$$

IV. CONCLUSION

In this paper, we presented one of the novel tools used in MSPC, which is the Hotelling's T^2 statistic. We also reviewed different popular Hotelling T^2 statistic decomposition techniques. By making use of real life dataset obtained from a cement production company, we were able to demonstrate the invariance property of the T^2 statistic which guarantees that each ordering of an observation vector will produce the same overall T^2 value.

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Biography

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APPENDIX 1
Boiler Temperature Data

Periods of Observations	X_1	X_2	X_3
1	507	516	527
2	512	513	533
3	520	512	537
4	520	514	538
5	530	515	542
6	528	516	541
7	522	513	537
8	527	509	537
9	533	514	528
10	530	512	538
11	530	512	541
12	527	513	541
13	529	514	542
14	522	509	539
15	532	515	545
16	531	514	543
17	535	514	542
18	516	515	537
19	514	510	532
20	536	512	540
21	522	514	540
22	520	514	540
23	526	517	546
24	527	514	543
25	529	518	544
Total	13125	12839	13473
Average	525	513.36	538.92

APPENDIX 2
Computation of Hotelling's T^2

Periods of Observations	Hotelling's T^2 Value	F- Critical Value
1	10.6950**	10.38
2	3.1972	
3	0.9558	
4	0.5567	
5	0.8560	
6	1.3567	
7	0.2384	
8	4.4559	
9	15.7660**	
10	1.2997	
11	1.1087	
12	0.3429	
13	0.4584	
14	5.3476	
15	1.7978	
16	0.8790	
17	1.9216	
18	2.0612	
19	4.6490	
20	3.2212	
21	0.4959	
22	1.0516	
23	4.1616	
24	0.8069	
25	4.3591	