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STOCHASTIC ANALYSIS OF A TWO SPECIES MODEL WITH COMMENSALISM

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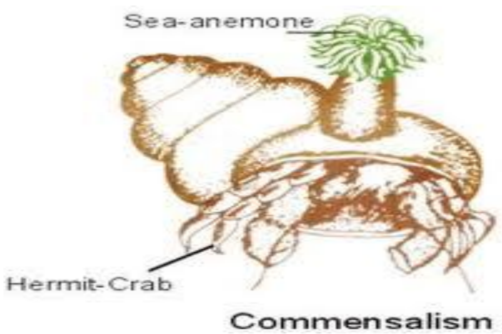
Abstract: This paper deals with an exploration on mathematical model of commensal and host species, in the presence of randomly fluctuating driving forces on the growth of the species at time 't' of a conventional eco system. The model consists of a commensal (S_1), a host (S_2) that benefit S_1 without getting effected either positively or adversely. The model is described by a pair of non-linear differential equations. Stochastic stability, in terms of the variances of the populations of the given system is derived using technique of Fourier transform.

Key words: Commensal, Stable, Fourier transform, host, stochastic stability, variance

I. INTRODUCTION

Mathematical modelling raised to the peak in recent years and spread to all branches of life sciences and fascinated the awareness of every one. Mathematical modeling of ecological unit was started by Lotka [1] and Volterra [2] and later several mathematicians and ecologists Meyer [3], Kushing [4], kapur [5-6] contributed to the growth of this area of acquaintance. The Ecological dealings can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and Parasitism. Competetive eco-systems with two species and three species were investigated by Srinivas [7]. Lakshmi narayan et.al [8] studied Prey-Predator ecological models with partial cover for the prey and alternate food for the predator. Archana Reddy [9] and Bhaskara Rama Sharma [10] investigated diverse problems related to two species competetive system with time delay. Later Phanikumar [11] investigated some mathematical models of ecological commensilism. More recently the criteria for a four species eco system were discussed at length by authors Hari Prasad et.al [12-14] and the authors [15-17] investigated the stability of three species and four species with stage structure, optimal harvesting policy and stochasticity. The present investigation is a study of stochastic stability of commensalism between two species and which is in terms of the variances of the populations of the given system. Hari prasad et.al [18], Kar et.al [19], Carletti [20] and Nisbet [21] inspired us to consider to do the present investigation is on the analytical and numerical approach of a two species eco system. Commensalism is a syn biotic interaction between two populations where one population (S_1) is benefited by the other population (S_2), while the other population (S_2) is neither harmed nor benefited due to interaction with the population (S_1).The benefited population (S_1) is called the commensal and the other population (S_2) is called the host. Some real life examples with photographs are given below

1) The clown fish shelters among the tentacles of the sea anemone, while the sea anemone is not affected.



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2) Epiphytes are small green plants found growing on other plants for space only. They absorb water and minerals from the atmosphere by their hygroscopic roots and prepare their own food. The plants are not harmed in anyway.

3) A Squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

II. STOCHASTIC MODEL

The model equations for two species commensal-host are given by the following system of non linear differential equations using the following details. $x(t)$:The population strength of commensal species (S_1) ; $y(t)$: The population strength of host species (S_2); t : The time instant; a_1 and a_2 : growth rates of commensal and host species respectively; a_{11} : self inhibition coefficient of (S_1) ; a_{22} : self inhibition coefficient of (S_2) ; a_{12} : commensal coefficient of (S_1) due to (S_2);

$$\frac{dx}{dt} = a_1 x - a_{11} x^2 + a_{12} xy + \eta_1 \varphi_1(t) \tag{2.1}$$

$$\frac{dy}{dt} = a_2 y - a_{22} y^2 + \eta_2 \varphi_2(t) \tag{2.2}$$

η_1 and η_2 are real constants and $\varphi(t) = [\varphi_1(t), \varphi_2(t)]$ is a two dimensional Gaussian white noise process agreeable

$$E[\varphi_i(t)] = 0;\ i = 1,2 \tag{2.3}$$

$$E[\varphi_i(t)\varphi_j(t')] = \delta_{ij}\delta(t - t');\ i = j = 1,2 \tag{2.4}$$

where δ_{ij} and δ are Kronecker and Dirac delta functions respectively.

Let $$x(t) = u_1(t) + S^*;\ y(t) = u_2(t) + P^*; \tag{2.5}$$

$$\frac{dx}{dt} = \frac{du_1(t)}{dt};\ \frac{dy}{dt} = \frac{du_2(t)}{dt};$$

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Using (2.5), equation (2.1) becomes

$$\frac{du_1(t)}{dt} = a_1u_1(t) + a_1S^* - a_{11}u_1^{\,2}(t) - a_{11}(S^*)^2 - 2a_{11}u_1(t)S^*$$
$$+a_{12}u_1(t)u_2(t) + a_{12}S^*P^* + a_{12}u_1(t)P^* + a_{12}u_2(t)S^* + \eta_1\varphi_1(t) \qquad (2.6)$$

The linear part of (2.6) is $\frac{du_1(t)}{dt} = -a_{11}u_1(t)S^* + a_{12}u_2(t)S^* + \eta_1\varphi_1(t)$ (2.7)

Again using (2.5) equation (2.2) becomes

$$\frac{du_2(t)}{dt} = a_2u_2(t) + a_2P^* - a_{22}u_2^{\,2}(t) - a_{22}(P^*)^2 - 2a_{22}u_2(t)P^* + \eta_2\varphi_2(t) \qquad (2.8)$$

The linear part of (2.8) is $\frac{du_2}{dt} = -a_{22}u_2(t)P^* + \eta_2\varphi_2(t)$ (2.9)

Taking the Fourier transform on both sides of (2.7), (2.9) we get,

$$\eta_1\tilde{\varphi}_1(\omega) = \left(i\omega + a_{11}S^*\right)\tilde{u}_1(\omega) - a_{12}S^*\tilde{u}_2(\omega) \qquad (2.10)$$

$$\eta_2\tilde{\varphi}_2(\omega) = \left(i\omega + a_{22}P^*\right)\tilde{u}_2(\omega) \qquad (2.11)$$

The matrix form of (2.10),(2.11) is $M(\omega)\tilde{u}(\omega) = \tilde{\varphi}(\omega)$ (2.12)

where

$$M(\omega) = \begin{pmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{pmatrix};\ \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \end{bmatrix};\ \tilde{\varphi}(\omega) = \begin{bmatrix} \tilde{\varphi}_1(\omega) \\ \tilde{\varphi}_2(\omega) \end{bmatrix};$$

$$A(\omega) = i\omega + a_{11}S^*;\ B(\omega) = -a_{12}S^*;\ C(\omega) = 0;\ D(\omega) = i\omega + a_{22}P^* \qquad (2.13)$$

Eqn.(2.12) can also be written as $\tilde{u}(\omega) = \left[M(\omega)\right]^{-1}\tilde{\varphi}(\omega)$

Let $\left[M(\omega)\right]^{-1} = K(\omega)$, therefore, $\tilde{u}(\omega) = K(\omega)\tilde{\varphi}(\omega)$ (2.14)

where $K(\omega) = \begin{bmatrix} \frac{D(\omega)}{|M(\omega)|} & -\frac{B(\omega)}{|M(\omega)|} \\ -\frac{C(\omega)}{|M(\omega)|} & \frac{A(\omega)}{|M(\omega)|} \end{bmatrix}$ (2.15)

if the function $Y(t)$ has a zero mean value , then the fluctuation intensity (variance) of it's components in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$

where $S_Y(\omega)$ is spectral density of Y and is defined as

$$S_Y(\omega) = \lim_{\tilde{T}\to\infty}\frac{\left|\tilde{Y}(\omega)\right|^2}{\tilde{T}} \qquad (2.16)$$

If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)e^{i\omega\tau}d\omega \qquad (2.17)$$

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^{\ 2} = C_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega \qquad (2.18)$$

and the auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} \qquad (2.19)$$

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For a Gaussian white noise process, it is

$$S_{\xi_i\xi_j}(\omega) = \lim_{\hat{T}\to+\infty} \frac{E\left[\tilde{\varphi}_i(\omega)\tilde{\varphi}_j(\omega)\right]}{\hat{T}}$$

$$= \lim_{\hat{T}\to+\infty} \frac{1}{\hat{T}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} E\left[\tilde{\varphi}_i(t)\tilde{\varphi}_j(t')\right] e^{-i\omega(t-t')} dt\, dt'$$

$$= \delta_{ij} \tag{2.20}$$

From (2.14), we have $\tilde{u}_i(\omega) = \sum_{j=1}^{2} K_{ij}(\omega)\tilde{\varphi}_j(\omega); i = 1,2$ (2.21)

From (2.16) we have $S_{u_i}(\omega) = \sum_{j=1}^{2} \eta_j \left|K_{ij}(\omega)\right|^2 \; ; i = 1,2$ (2.22)

Hence by (2.18) and (2.22), the intensities of fluctuations in the variable u_i ; $i = 1,2$ are given by

$$\sigma_{u_i}^{\;2} = \frac{1}{2\pi} \sum_{j=1}^{2} \int_{-\infty}^{\infty} \eta_j \left|K_{ij}(\omega)\right|^2 d\omega; \; i = 1,2 \tag{2.23}$$

and by (2.15), we obtain

$$\sigma_{u_1}^{\;2} = \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} \alpha_1 \left|\frac{D(\omega)}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left|\frac{B(\omega)}{|M(\omega)|}\right|^2 d\omega \right\}$$

$$\sigma_{u_2}^{\;2} = \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} \alpha_1 \left|\frac{A(\omega)}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left|\frac{C(\omega)}{|M(\omega)|}\right|^2 d\omega \right\} \tag{2.24}$$

Where $|M(\omega)| = |R(\omega)| + i\,|I(\omega)|$ (2.25)

Real part of $|M(\omega)| = R^2(\omega) = \left[-\omega^2 + a_{11}a_{22}S^*P^*\right]^2$ (2.26)

Imaginary part of $|M(\omega)| = I^2(\omega) = (\omega a_{22}P^* + \omega a_{11}S^*)^2$ (2.27)

Finally from (2.13) we get $|A(\omega)|^2 = \omega^2 + (a_{11}S^*)^2$;

$|B(\omega)|^2 = (a_{12}S^*)^2 ; |C(\omega)|^2 = 0 ; |D(\omega)|^2 = \omega^2 + (a_{22}P^*)^2$ (2.28)

By substitution of (2.25), (2.13) in (2.24), we get,

$$\sigma_{u_1}^{\;2} = \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} \left[\eta_1\left\{\omega^2 + (a_{22}P^*)^2\right\} + \eta_2(a_{12}S^*)^2\right] d\omega \right\} \tag{2.29}$$

$$\sigma_{u_2}^{\;2} = \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} \left[\eta_1\left\{\omega^2 + (a_{11}S^*)^2\right\} + \eta_2(0)^2\right] d\omega \right\} \tag{2.30}$$

If we are interested in the dynamics of system (2.1)-(2.2) with either $\eta_1 = 0$ or $\eta_2 = 0$ then the population variances are

If $\eta_1 = 0$,then $\sigma_{u_1}^{\;2} = \frac{\eta_2(a_{12}S^*)^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} d\omega$ (2.31)

$$\sigma_{u_2}^{\;2} = 0 \tag{2.32}$$

If $\eta_2 = 0$, then $\sigma_{u_1}^{\;2} = \frac{\eta_1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega)+I^2(\omega)} \left[\omega^2 + (a_{22}P^*)^2\right] d\omega$ (2.33)

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$$\sigma_{u_2}{}^2 = \frac{\eta_1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\omega^2+(\alpha_{11}S^*)^2\right]d\omega \tag{2.34}$$

The expressions in (2.24) give three variances of the populations. The amalgamations over the real line can be appraised which gives the variances of the populations.

III. COMPUTER SIMULATION

For support of our earlier discussed analytical results, we here would like to present some numerical replications with the help of MATLAB 7.3 software package.
Example (1) a1=2.5;a11=0.1;a12=1.5;omga=3.5; a2=1.5;a22=0.8;gama=5.5;

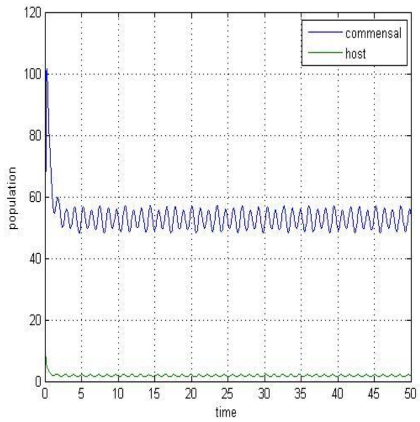

Fig. 3.1(a)

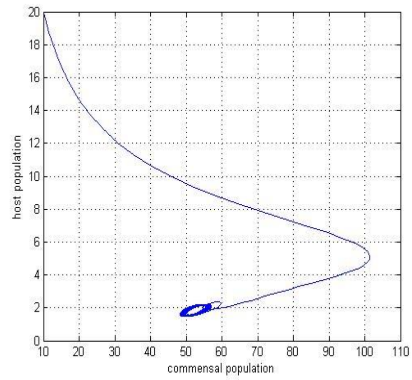

Fig. 3.1(b)

Figures 3.1(a) and 3.1(b) shows that the variation of population against time under random environmental noise, population oscillation gives periodic against time and phase portrait diagram between commensal population and host population respectively with initial values $x=10$, y=20

Example (2) a1=0.5;a11=0.1;a12=1.5;omga=3.5;a2=0.5;a22=0.5;gama=7.5;

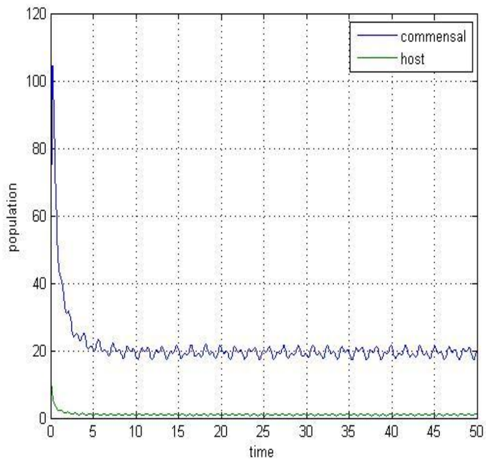

Fig. 3.2(a)

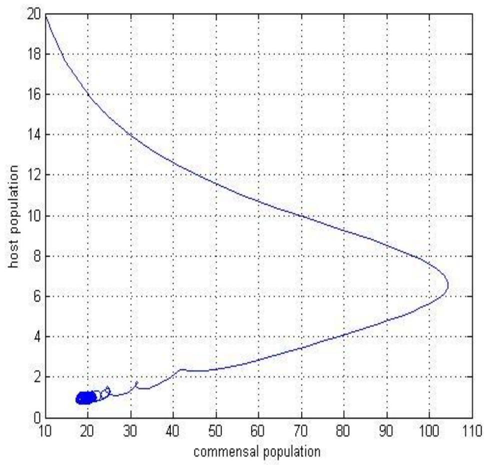

Fig. 3.2(b)

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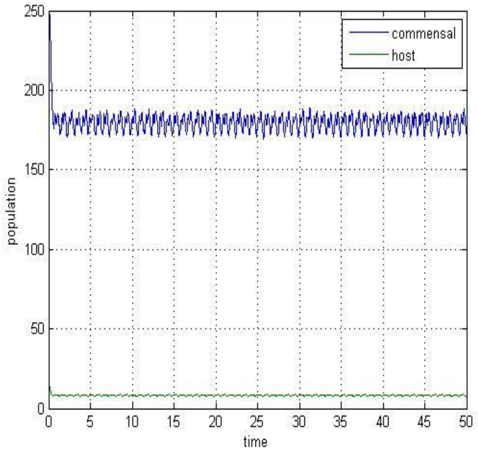
Figures 3.2(a) and 3.2(b) shows that the variation of population against time under random environmental noise, population oscillation gives periodic against time and phase portrait diagram between commensal population and host population respectively with initial values x=10, y=20

Example (3): a1=5.5;a11=0.1;a12=1.5;omga=3.5; a2=2.5;a22=0.5;gama=7.5;



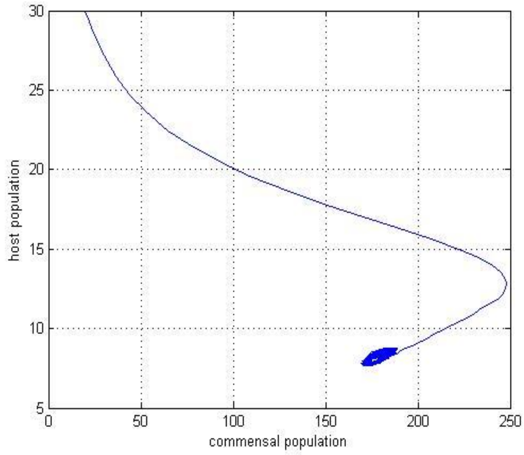
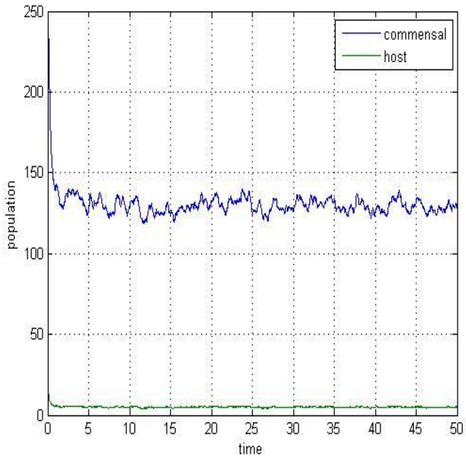


Fig. 3.3(a) Fig. 3.3(b)

Figures 3.3(a) and 3.3(b) shows that the variation of population against time under random environmental noise, population oscillation gives periodic against time and phase portrait diagram between commensal population and host population respectively with initial values $x = 20$, y=30

Example (4): a1=5.5;a11=0.1;a12=1.5;omga=3.5;a2=2.5;a22=0.5;gama=7.5;



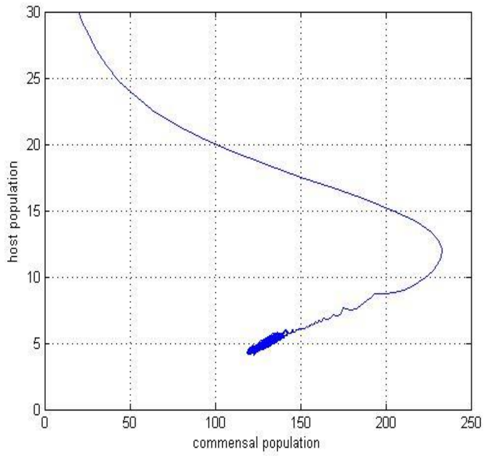


Fig. 3.4(a) Fig. 3.4(b)

Figures 3.4(a) and 3.4(b) shows that the variation of population against time under random environmental noise, population oscillation gives periodic against time and phase portrait diagram between commensal population and host population respectively with initial values $x = 20$, y=30

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IV. CONCLUDING REMARKS

In this paper, a model of a distinctive two species commensal, host system in which stochastic term was invented and investigated the effect of environmental fluctuations around the positive equilibrium due to additive white noise. The population variances are computed and analyzed for stability using Matlab. Further for stochastic system, population variances have a great role to analyze the stability of the system. The conclusion is that the noise on the equation results in an immense variances of oscillations around the equilibrium point which propose that our system is periodic with respect to a noisy atmosphere. Numerical replications reveal that the trajectories of the system oscillate arbitrarily with remarkable variance of amplitudes with the increasing value of the strength of noises initially but ultimately fluctuating which are viewed in figures 3.1(a,b)-3.4(a,b). Hence we conclude that inclusion of stochastic perturbation create a significant change in intensity in our considering commensal host scheme due to change of responsive parameters which causes large environmental fluctuations.

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