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Cone Metric Spaces and Common Fixed Point Theorems for Generalized Multivalued Mappings

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Abstract Let *P* be a sub set of Banach space *E* and *P* is normal and regular cone on *E*; we generalize and obtain some sufficient conditions for the existence of common fixed point of multivalued mappings satisfying contractive type conditions in cone metric spaces. Our results unify, generalize and complement the comparable results from the current literature.

Key Words: Common fixed point, Cone metric spaces, multivalued mapping, Normal cone and non- normal cone

I. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays a basic role in applications of various braches of mathematics, from elementary calculus and linear algebra to topology and analysis. Fixed point theory is not restricted to mathematics and this theory has many applications in other disciplines. This theory is closely related to game theory, military, economics, statistics and medicine. Much work has been done involving fixed points for multivalued contractions and none expansive maps using the Hausdorff metric was initiated by Markin [1].Later, an interesting and rich fixed point theory for such maps was developed. Nadler Jr. [2] has proved valtivalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space has a fixed point(see also[3],[4],[5],[6],[7]and[8]).

Quiet recently, Huang and Zhang [9] introduced the concept of cone metric space, replacing the set of positive real numbers by an ordered Banach space. He also gave the condition in the setting of cone metric space. These authors also studied the strong convergence to a fixed point with contractive constant in metric space and introduce the corresponding notion of completeness. Subsequently many authors have generalized the results of Huang and Zhang [9] and have studied fixed point theorems for normal and non-normal cone (see [10]). S. Hoon .Cho and Mi Sun Kim [11] have proved certain fixed point theorems using Multivalued mapping in the setting of contractive constant in metric spaces and also S. Hoon Cho and J.S. Base [12] proved fixed point theorems for multivalued mappings in cone metric spaces. R. C. Dimri, Amit Singh and Sandeep Bhatt[13] also proved common fixed point theorems for two multivalued maps in complete metric spaces with normal constant M=1. Further, Mujahid Abbas, B.E.Rhoades and Talat Nazir [14] obtained and generalized sufficient conditions for the existence of common fixed points of multivalued mapping satisfying contractive conditions in non-normal cone

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metric space.Wardowski [15] introduced the concept of multivalued contractions in cone metric spaces and, using the notion of normal cones, obtained fixed point theorems for such mappings.
The purpose of this paper is to prove some common fixed point results for multivalued mappings taking normal and non-normal in cone metric spaces. Our results extend and unify various comparable results in literature [16], [17], and [18]

Definition 1.1 [9] Let E be a real Branch space and P a subset of E. Then P is called a cone if it is satisfied the following conditions,
(I) P is closed, non-empty and $P \neq \{0\}$;
(II) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers $a, b \in R$;
(III) $x \in P \; and - x \in P \; => \; x = 0.$

For a given cone$P \subset E$, we define a Partial ordering $\leq$ on E with respect to P $by \; x \leq y$ if and only if $y - x \in P.$ we shall write $x \ll y$ if $y - x \in$ int P, int P denotes the interior of P. The cone P is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Definition 1.2 [9]: Let X be a non- empty set. Suppose that the mapping $d: X \, x \, X \rightarrow E$ satisfies,
(I) 0< $d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(II) $d(x, y) = d(y, x)$ for all x , $y \in X$;
(III) $d(x, y) =(x, z) + (y, z)$ for all $x, y \in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.3 [9]: Let $E = R^2$, $P = \{(x, y) \in E: x, y \geq 0\}$, $X = R$ and $d: X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, \propto |x - y|)$ where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space.
Example 1.4 Let E= l^1, P = $\left\{\{x_n\}n \geq 1 \in E: x_n \geq 0, for \; all \; n\right\}$ (X, d) a metric space and $d: X \times X \rightarrow E$ defined by $d(x, y) = \left\{\frac{d(x,y)}{2^n}\right\}$ $n \geq 1$. Then (X, d) is a cone metric space.

Definition 1.5[9]: Let (X, d) be a cone metric space. Let $\{\boldsymbol{x}_n\}_{n\geq1}$ be a sequence in X and $\boldsymbol{x} \in X$. Then
(I) $\{ x_n \}_{n\geq1}$ converges to x whenever for every c $\in$E with $0 \ll c$ there is a natural number N such that $d(\boldsymbol{x}_n, x) \ll c$ for all $n \geq N$
(II) $\{ \boldsymbol{x}_n\}_{n\geq1}$ is said to Cauchy sequence for every c $\in$ E with $0 \ll$ c there is a natural number N such that $d(\boldsymbol{x}_n, \boldsymbol{x}_m) \ll$c $for \; all \; n \; m \geq N.$
(III) (X, d) is called a complete cone metric space, if every Cauchy sequence is convergent in X.

Let (X, d) be a cone metric space. We denote by $CB(X)$ the family of non-empty closed bounded sub set X. Let $H(-,-)$Hausdorff distance on $CB(X)$ $i.e.$ for $A, B \in CB(X), H(A, B) = \max\{sup_{a\in A} d(a, B), sup_{b\in B} \, d(A, b)\}$ where $d(a, B) inf\{d(a, b): b \in B\}$ is the distance from the point a to the sub set B. An element $x \in X$ is said to be a fixed point of a multivalued mapping$T: x \rightarrow 2^x$, if $x \in T$(x).

Lemma 1.6[19] A function F: P→P is called $\leq$ increasing, if for each x, y $\in$ P, $\boldsymbol{x} \ll y$ if and only if $f(x) \ll f(y)$.
Let $F: P \rightarrow P$ be a function such that
(F_1) $F(t)$= 0 if and only if t = 0;
(F_2) F is $\ll$ -increasing;
(F_3) F is surjective.

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We denote by $\Im(P,P)$ the family of functions satisfying (F_1),$(F_2)$and (F_3).
Example 1.7 Let $F(t) = t$ for each $t \in P$. Then$\in \Im(P,P)$.

Lemma1.8 (I) Let $\varphi: P \to P$ be a mapping subadditive, then $F \in \Im(P,P)$ is sub additive.

(II) Let E be a Banach space. If $c_n \in E$ and $c_n \longrightarrow 0$, then for each $c \in \text{int}(P)$ there exist N such that $c_n \ll c$ *for all* $n > N$.

II. MAIN RESULTS

Theorem 2.1: Let (X, d) be a complete cone metric space and let $T, S: X \longrightarrow CB(X)$ be any two multivalued maps satisfying for each $x, y \in X$,

$$F[H(Tx,Sy)] \leq \alpha\,[F\{d(x,y) + d\,(Tx,Sy)\}] + \beta\,[F\{d(x,Tx) + d(y,Sy)\}] + \gamma[F\{d(x,Sy) + d(y,Tx)\}] \qquad (2.1.1)$$

for all $\boldsymbol{x}, y \in X, \alpha+\beta+\gamma < \frac{1}{2}$, with for some $\alpha,\beta,\gamma \in \left[0,\frac{1}{2}\right]$. And $F \in \Im(P,P)$ *such that* F is sub additive and $\lim_{n\to\infty} F(c_n) = 0$ *i.e* $\lim_{n\to\infty}(c_n) = 0$ .Then T and S have a common fixed point.

Proof: For every $x_0 \in X$ *and* $n \geq 1$, $x_1 \in T(x_0)$ and $x_{2n+1} \in T(x_{2n})$ *i.e.*$x_{2n+1} = Tx_{2n}$

Similarly we can have $x_2 \in T(x_1)$ *and* $x_{2n+2} \in Sx_{2n+1}$ *i.e* $x_{2n+2} = Sx_{2n+1}$. Then we have

$$F[d(x_{2n+1},x_{2n})] \leq F\left[H\left(Tx_{2n,}Sx_{2n-1}\right)\right]$$

$$\leq \alpha\left[F\left\{d\left(x_{2n,}x_{2n-1}\right) + d(\,Tx_{2n},Sx_{2n-1}\right\}\right] + \beta\left[F\left\{d(x_{2n},Tx_{2n}) + d(x_{2n-1,}Sx_{2n-1}\right\}\right] + \gamma\left[F\left\{d\left(x_{2n,}\,S\,x_{2n-1}\right) + d(x_{2n-1},Tx_{2n}\right\}\right]$$

$$\leq \alpha\left[F\left\{d\left(x_{2n,}x_{2n-1}\right) + d(\,x_{2n+1},x_{2n}\right\}\right] + \beta\left[F\left\{d(x_{2n},x_{2n+1}) + d(x_{2n-1,}x_{2n}\right\}\right] + \gamma\left[F\left\{d\left(x_{2n,}\;x_{2n}\right) + d(x_{2n-1},x_{2n+1}\right\}\right]$$

$$\leq \alpha\left[F\left\{d\left(x_{2n,}x_{2n-1}\right) + d(\,x_{2n+1},x_{2n}\right\}\right] + \beta\left[F\left\{d(x_{2n},x_{2n+1}) + d(x_{2n-1,}x_{2n}\right\}\right] + \gamma[F\{d(x_{2n-1},x_{2n+1})\}]$$

$$\leq \alpha\left[F\left\{d\left(x_{2n,}x_{2n-1}\right) + d(\,x_{2n+1},x_{2n}\right\}\right] + \beta\left[F\left\{d(x_{2n},x_{2n+1}) + d(x_{2n-1,}x_{2n}\right\}\right] + \gamma[F\{d(x_{2n},x_{2n+1}) + d(x_{2n-1},x_{2n})\}]$$

$$\leq (\alpha+\beta+\gamma)\,F\,[d(x_{2n},x_{2n+1}) + d(x_{2n},x_{2n-1})]$$

$$\Rightarrow F[d(x_{2n+1},x_{2n})] \leq \left(\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta+\gamma)}\right) F[d(x_{2n},x_{2n-1})] \qquad (2.1.2)$$

Where $\left(\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta+\gamma)}\right) = \text{L}$

Hence $F[d(x_{2n+1},x_{2n})] = L^n F[d(x_1,x_0)]$ (2.1.3)

For $n > m$, we have that

$$F[d(x_{2n},x_{2m})] \leq F[d(x_{2n},x_{2n-1}) + d(x_{2n-1},x_{2n-2}) \ldots\ldots\ldots\ldots + d(x_{2m+1},x_{2m})]$$
$$\leq F[L^{2n-1} + L^{2n-2} + L^{2.n-3} + \cdots\ldots\ldots\ldots\ldots + L^{2m}]\; d(x_1,x_0)$$
$$\leq \frac{L^{2m}}{1-L} F[d(x_1,x_0)] \qquad (2.1.4)$$

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Hence $\lim_{n\to\infty} d(x_{2n}, x_{2m}) = 0$ by (2.1). Applying Lemma 1.8 (ii), $\{x_{2n}\}$ is Cauchy sequence in (X, d), for $n > m$. Since (X, d) be a complete cone metric space, there exist $\boldsymbol{u} \in X$suchthat $\lim_{n\to\infty}(x_{2n}) = \boldsymbol{u}$. i.e. $x_{2n}\longrightarrow\boldsymbol{u}$.Let $c \in int(P)$be given. We can choose a natural number N_1such that $d(x_{2n}, u) \ll F^{-1}\left(\frac{c(1-L)}{3}\right), forall\ n{>}N_1$.By (F_2) and (F_3), $F\left(d\left(x_{2n,}u\right)\right) \ll \frac{c(1-L)}{3}$ for all $n > N_1$.

Then we have,

$$\begin{aligned}
F[d(Tu,u)] &\le F\left[H\left(Tx_{2n,}Tu\right) + d(Tx_{2n}, u)\right] \\
&\le \alpha[F\{d(x_{2n}, u) + d(Tx_{2n}, Tu)\}]{+}\beta[F\{d(x_{2n}, Tx_{2n}) + d(u, Tu\}] \\
&\quad + \gamma[F\{d(x_{2n}, Tu) + d(Tx_{2n}, u\}]{+}F[d(x_{2n+1}, u)] \\
&\le \alpha[F\{d(x_{2n}, u) + d(x_{2n+1}, Tu)\}]{+}\beta[F\{d(x_{2n}, x_{2n+1}) + d(u, Tu\}] \\
&\quad + \gamma[F\{d(x_{2n}, Tu) + d(x_{2n+1}, u\}]{+}F[d(x_{2n+1}, u)] \\
&\le \alpha[F\{d(x_{2n}, Tu) + d(u, Tu)\}]{+}\beta[F\{d(x_{2n}, x_{2n+1}) + d(u, Tu\} + d(x_{2n+1}, u)] \\
&\quad + \gamma[F\{d(x_{2n}, u) + d(x_{2n+1}, u) + d(Tx_{2n}, u)\}]{+}\, d(x_{2n+1}, u)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (1-k)F\,[d(Tu,u)] &\le k[F\{d(x_{2n}, u)\}] + k[F\{d(x_{2n+1}, u\}]_{+}[F\{\, d(x_{2n+1}, u)\}] \\
&\le [F\{d(x_{2n}, u)\}] + [F\{d(x_{2n+1}, u\}]_{+}[F\,\{d(x_{2n+1}, u)\}]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow F\,[d(Tu,u)] &\le \frac{F(d(x_{2n},u)+d(x_{2n+1},u)+d(x_{2n+1},u))}{1-K} \qquad (2.1.5) \\
&\le \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c, \text{ for all } n \ge N_1, \text{ thus } F\,[d(Tu,u)] \ll \frac{c}{m} \text{ for all } m \ge 1,
\end{aligned}$$

We get $\frac{c}{m} - F[d(Tu,u)] \in P$, and as $m\longrightarrow\infty$ we get$\frac{c}{m} \to 0$, and P is closed, $-F[d(Tu,u)] \in P$.Hence $F[d(Tu,u)] = 0$. By $(F_1), d(Tu,u) = 0$ and so $\boldsymbol{u} \in Tu$ .Now if $\boldsymbol{v}$ is another fixed point of T. Then from 3.1 we have $F[d(u,v)]$ =0. Hence $F[d(u,v)] \in -P$, and $F[d(u,v)] = 0$.By(F_1), $d(u,v)$= 0 and $u \in Tv$. Hence $\boldsymbol{u} \in Tu = \boldsymbol{u} = u \in Tv$. Therefore, $\boldsymbol{u}$ is the fixed point of T. Similarly, it can be established that $\boldsymbol{u} \in Tu = \boldsymbol{u} = \boldsymbol{u} \in Su$. Thus $\boldsymbol{u}$ is the common fixed points T and S.

Corollary: 2.2 Let $(X,\ d)$ be a complete cone metric space and let T, S : X$\longrightarrow$ CB (X) be any two multivalued maps satisfying for each $x, y \in X$,

$$F[H(Tx,Sy)\,] \le\ \alpha\,[F\{d(x,y) + d\,(Tx,Sy)\}] +\ \beta\,[F\{d(x,Tx) + d(y,Sy)\}] \qquad (2.1.6)$$

for all $\boldsymbol{x}$, $y \in X$, $where\ \alpha,\beta, \in \left[0,\frac{1}{2}\right]$. And F$\in$ $\mathfrak{J}(P,P)\ such\ that$F is sub additive and $\lim_{n\to\infty}F(c_n) = 0\ i.e\ \lim_{n\to\infty}(c_n) = 0$ .Then T and S have a common fixed points in X.

Proof: If we take $\gamma = 0$ in theorem (2.1), then we get the above result.

Corollary: 2.3 Let (X,d) be a complete cone metric space and let T, S: X$\longrightarrow$ CB(X) be any two multivalued maps satisfying for each $x, y \in X$,

$$F[H(Tx,Sy)] \le\ \beta\,[F\{d(x,Tx) + d(y,Sy)\}] \qquad (2.1.7)$$

for all $\boldsymbol{x}$,$y \in X$, $where\ ,\beta, \in \left[0,\frac{1}{2}\right]$. And $F\ \in\ \mathfrak{J}(P,P)\ such\ that\ \ F$ is sub additive and $\lim_{n\to\infty}F(c_n) = 0\ i.e\ \lim_{n\to\infty}(c_n) = 0$ .Then T and S have common fixed points in X.

Proof: If we take $\alpha = 0$ and $\gamma = 0$in previous theorem 3.1, then we get the above result.

Remark 2.4 If we take $\alpha = 0$ previous theorem 3.1then we get the result of

$$F[H(Tx,Sy)] \le \beta\,[F\{d(x,Tx) + d(y,Sy)\}] + \gamma[F\{d(x,Sy) +\ d(y,Tx)\}]\,,$$

WhereT, S : $X \longrightarrow CB(X)$ be any two multivalued maps, for all x, y $\in$ X, $\beta + \gamma < \frac{1}{2}$, with for some $\beta, \gamma \in \left[0,\frac{1}{2}\right]$. And F$\in$ $\mathfrak{J}(P,P)\ such\ that$ F is sub additive and $\lim_{n\to\infty}F(c_n) = 0\ i.e\ \lim_{n\to\infty}(c_n) = 0$.

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Theorem 2.5 Let (X,d) be a complete cone metric space and let $T, S : X \longrightarrow CB(X)$ be any two multivalued maps satisfying for each $x, y \in X$,

$$F[H(Tx,Sy)] \leq r\, max\ [F\{d\,(x,y)d(x,Tx)d(y,Sy)\}] \quad (2.1.8)$$

For all $\boldsymbol{x}$, $y \in X, and\ r \in [0,1]$. And $F \in \mathfrak{I}(P,P)\ such\ that\ F$ is sub additive and $im_{n\to\infty}F(c_n) = 0\ i.e\ \lim_{n\to\infty}(c_n) = 0$ .Then T and S have a common fixed point.

Proof: For every $x_0\ \in X\ and\ n \geq 1,\ x_1\ \in T(x_0)$ and $x_{2n+1} \in T(x_{2n})\ i.e. x_{2n+1}\ = Tx_{2n}$

Then we have

$$\begin{aligned} F[d(x_{2n+1},x_{2n})] &\leq F\left[H\left(Tx_{2n,}Sx_{2n-1}\right)\right] \\ &\leq r\,max\left[F\left\{d\left(x_{2n,},x_{2n-1}\right)\ d(x_{2n},Tx_{2n})\ d(x_{2n-1},Sx_{2n-1})\right\}\right] \\ &\leq r\,max\left[F\left\{d\left(x_{2n,},x_{2n-1}\right)\ d(x_{2n},x_{2n+1})\ d(x_{2n-1},x_{2n})\right\}\right] \\ &\leq r[F\{d(x_{2n-1}\,,x_{2n})\}] \\ \Rightarrow\ F[d(x_{2n+1},x_{2n})] &\leq r^n[F\{d(x_1\,,x_0)\}] \quad (2.1.9) \end{aligned}$$

So, for $n > m$, we have

$$\begin{aligned} F[d(x_{2n},x_{2m})] &\leq F[d(x_{2n},x_{2n-1}) + d(x_{2n-1},x_{2n-2}) + \cdots \dots \dots \dots \dots + d(x_{2m+1},x_{2m})] \\ &\leq\ [r^{2n-1} + r^{2n-2} + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots + r^{2m}]Fd\,(x_1,x_0) \\ &\leq \frac{r^{2m}}{1-r}F[d(x_1,x_0)] \quad (2.1.10) \end{aligned}$$

Hence $\lim_{n\,,m\to\infty}\ d(x_{2n},x_{2m}) = 0$ by (2.5). Applying lemma 1.8 (ii), $\{x_{2n}\}$ is Cauchy sequence in (X,d), for $n > m$. Since (X,d) be a complete cone metric space, there exist $\boldsymbol{u} \in X$ such that $\lim_{n\to\infty} x_{2n} = \boldsymbol{u}. i.e\ x_{2n} = u$. Let $c \in intP$ be given. Choose a natural number N_1 such that $d(x_{2n,}u) \ll F^{-1}\left(\frac{C}{3}\right)$ for all $n \geq N_1$. By $(F_2) and (F_3)$, $F[d(x_{2n},u)] \ll \frac{c}{3}$, for all $n \geq N_1$. Then we have

$$\begin{aligned} F[d(Tu\,,u)] &\leq\ F[H\{d(Tx_{2n},Tu) + d(Tx_{2n},u)\}] \\ &\leq r\,max[F\{d(x_{2n},u)d(x_{2n},Tx_{2n})d(u,Tu)\}]F\left[d\left(x_{2n+1,}u\right)\right] \\ &\leq r\,max[F\{d(x_{2n},u)d(x_{2n},x_{2n+1})d(u,Tu)\}] + F\left[d\left(x_{2n+1,}u\right)\right] \\ &\leq r\,max[F\{d(x_{2n},u)d(x_{2n},u)\ \} + F\{d(u,x_{2n+1})d(u,Tu)\}] \end{aligned}$$

$\Rightarrow F[d(Tu\,,u)] \leq c$, for all $n \geq N_1$. Thus $F[d(Tu\,,u)] \ll \frac{c}{m}, for\ all\ m \geq 1$. we get $\frac{c}{m} - F[d(Tu,u)] \in$P, and as $m \longrightarrow \infty$ we get $\frac{c}{m} \longrightarrow 0$.Since P is closed, $-F[d(Tu,u)] \in P.$

Hence $F[d(Tu\,,u)] = 0$. By (F_1), $d(Tu,u) = 0$.So, $u \in Tu\ i.e. u = Tu.$ Now, if v is another fixed point of T. Then from 2.5, we have $Fd(u,v) = 0$. Hence $F\ d(u,v) \in -P,$

Hence $F[d(u,v)] = 0$. By (F_1), $d(u,v) = 0$ and $v \in Tv$. Then $u \in Tu = u = v \in Tv$.

Therefore, $\boldsymbol{u}$ is the unique fixed point of T.

Similarly we can establish that, $u \in Tu = u = u \in Su$.Thus $\boldsymbol{u}$ is the common fixed point of T and S.

Corollary: 2.6. Let (X, *d)* be a complete cone metric space and let *T, S*: $X \longrightarrow CB(X)$ be any two multivalued maps satisfying for each $x, y \in X$,

$$F[H(Tx,Sy)] \leq\ k\ [F\{d(x,y)\}]\dots \quad (2.\ 1.11)$$

for all $\boldsymbol{x}, y \in X,\ where\ , k \in \left[0,\frac{1}{2}\right]$. And $F\ \in \mathfrak{I}(P,P)\ such\ that$ F is sub additive and $\lim_{n\to\infty}F(c_n) = 0\ i.e\ \lim_{n\to\infty}(c_n) = 0$. Then T and S have a common fixed point in X.

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Proof: The proof of the corollary immediately follow by taking $d(x, y)$ as maximum value in the previous theorems 2.5.

Note: 2.7 we have prove the above theorems in the setting P is a normal cone with the normal constant k

Theorem 2. 8. Let (X, d) be a complete cone metric space and P be a normal cone with normal constant$K$. Suppose $T, S: X \longrightarrow CB(X)$be any two multivalued maps satisfying each $X, F[H\{Tx, Sy\}] \leq r\, max[F\{d(x,y)d(x,Tx)d(y,Sy)d(x,Sy)d(Tx,y)\}] \ldots\ldots\ldots\ldots\ldots[12]$

$f$or all x ,y $\in$ X where $r \in [0,1]$.And $F \in \mathfrak{I}[P, P]$ such that F is sub additive and $\lim_{n\to\infty} F(c_n) = 0$. i.e. $\lim_{n\to\infty}(c_n) = 0$.Then T and S have a common fixed point in X.

Proof: For every $x_0 \in X$ and $n \geq 1$, $x_1 \in T(x_0)$and$x_{2n+1} \in T(x_{2n})$ $i.e. x_{2n+1} = Tx_{2n}$.

Similarly we can have $x_2 \in T(x_1)$ and $x_{2n+2} \in Sx_{2n+1}$ $i.e.\ x_{2n+2=} Sx_{2n+1}$.

Then we have

$$\begin{aligned}
F[d(x_{2n+1}x_{2n})] &\leq F[H(Tx_{2n}, Sx_{2n-1})] \\
&\leq r\, max\, \left[F\{d(x_{2n}, x_{2n-1})d(x_{2n}, Tx_{2n})d(x_{2n-1}Sx_{2n-1})d\left(x_{2n,}Sx_{2n-1}\right)d(Tx_{2n}, x_{2n-1})\}\right] \\
&\leq r\, max\, [F\{d(x_{2n}, x_{2n-1})d(x_{2n}, x_{2n+1})d(x_{2n-1}, x_{2n})d(x_{2n}, x_{2n})d(x_{2n+1}, x_{2n-1})\}] \\
&\leq r\, max\, [F\{d(x_{2n}, x_{2n-1})d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n-1})\}] \\
&\leq r\, max\, [F\{d(x_{2n}, x_{2n-1})d(x_{2n+1}, x_{2n-1})\}] \\
\Rightarrow F[d(x_{2n+1}, x_{2n})] &\leq r[F\{d(x_{2n}, x_{2n-1})\}],
\end{aligned}$$

Then we get

$$F\left[d\left(x_{2n+1,}x_{2n}\right)\right] \leq r^n[d(x_1, x_0)]. \tag{2.1.13}$$

So for n > m, we have

$$\begin{aligned}
F[d(x_{2n}, x_{2m})] &\leq F[d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + \cdots \ldots\ldots\ldots\ldots + d(x_{2m+1}, x_{2m})] \\
&\leq [r^{2n-1} + r^{2n-2} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + r^{2m}]Fd\,(x_1, x_0) \\
&\leq \frac{r^{2m}}{1-r} F[d(x_1, x_0)] \ldots
\end{aligned} \tag{2.1.14}$$

We get $F\|d(x_{2n}, x_{2m})\| \leq K\, \frac{r^{2m}}{1-r} F\, \|d(x_1, x_0)\|$. Hence $\lim_{n,m\to\infty} d(x_{2n}, x_{2m})$= 0,by 2.8. $i.e.$ $F[d(x_{2n}, x_{2m})] \longrightarrow 0$ as $n, m \to 0$.Applying lemma 1.8](ii), $\{x_{2n}\}$ is a Cauchy sequence in X. Since(X, d) be a complete cone metric space, there exist $u \in X$ such that$\lim_{n\to\infty} x_{2n=}u$. i.e. $x_{2n} \to u$ as $n \to \infty$.

Since $\quad F[d(Tu, u)] \leq F[H(Tx_{2n}, Tu) + d(Tx_{2n}, u)]$

$$\leq r\, max[F\{d(x_{2n}, u)\, d(x_{2n}, Tx_{2n})d(u, Tu)d(x_{2n}, Tu)d(Tx_{2n}, u)\}] + F[d(x_{2n+1}, u)]$$

$$\leq r\, max[F\{d(x_{2n}, u)d(x_{2n}, x_{2n+1})d(u, Tu)d(x_{2n}, Tu)d(x_{2n+1}, u)\}]$$

$$+ F[(x_{2n+1}, u)]$$

$$\leq r\, [F\{d(u, Tu)\}]$$

$F[\|d(Tu, u)\|] =$ K$[k\|d(u, Tu)\|] \longrightarrow 0$.Hence$F\|d(Tu, u)\|$= 0, by$(F_1)$,$\|d(Tu, u)\|$=0.

$\Rightarrow Tu = u$. So $\boldsymbol{u}$ is a fixed point of T. Now v is a another fixed point of T, then

$$\begin{aligned}
F[H(u, v)] &\leq F[H(Tu, Tv)] \\
&\leq r\, max[F\{d(u, v)d(u, Tu)d(v, Tv)d(u, Tv)d(Tu, v)\}]
\end{aligned}$$

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$\leq r\, max[F\{d(u,v)d(u,u)d(v,v)d(u,v)d(u,v)\}]$
$\leq K\ [F\{d(u,v)\}] \longrightarrow 0$. Hence $F\|d(u,v)\| = 0$ by (F1),$\|d(u,v)\|$ =0.

And $u = v$. Therefore, u is the unique fixed point of T in X.
Similarly, it can be establish thatSu = u.Hence $Tu = u = Su$. Thus u is common fixed points of T and S in$X$.

Theorem 2.9 Let (X,d) be a complete cone metric space and P be a normal cone with normal constant$K$. Let T,S : X⟶ $CB(X)$ be any two multivalued maps satisfying for each $x,\ y \in X$,

$$F[H(Tx,Sy)\ ] \leq \alpha\left[F\left\{\frac{d(x,Tx)d(y,Sy)d(x,Sy)+d(x,y)d(y,Tx)d(x,Tx)}{1+d(y,Tx)d(y,Sy)}\right\}\right]$$
$$+\beta[F\{d(x,Tx)+d(y,Sy)\}]+\gamma[F\{d(x,Sy)+d(y,Tx)\}]$$
$$+\delta[F\{d(x,y)\}] \tag{2.1.15}$$

$f$or all $\boldsymbol{x}$,$y \in X$ where $r \in [0,1]$.And $F \in \Im[P\ ,P]$ such that F is sub additive and $\lim_{n\to\infty} F(c_n) = 0$. i.e. $\lim_{n\to\infty}(c_n) = 0$.Then T and S have a common fixed point in X.

Proof: For every $x_0 \in X\ and\ n \geq 1,\ x_1 \in T(x_0)andx_{2n+1} \in T(x_{2n})\ i.e. x_{2n+1} = Tx_{2n}$.
Similarly we can have $x_2 \in T(x_1)$ and $x_{2n+2} \in Sx_{2n+1}\ i.e.\ x_{2n+2=}\ Sx_{2n+1}$.

Then we have,

$$F[d(x_{2n+1}x_{2n})] \leq F[H(Tx_{2n},Sx_{2n-1})]$$
$$\leq\alpha\left[\frac{F\{d(x_{2n},Tx_{2n})d(x_{2n-1},Sx_{2n-1})d(x_{2n},Sx_{2n-1})+d(x_{2n},x_{2n-1})d(x_{2n-1},Tx_{2n})d(x_{2n},Tx_{2n})\}}{1+F\{d(x_{2n-1},Tx_{2n})d(x_{2n-1},Sx_{2n-1})\}}\right]$$
$$+\beta\ [F\{d(x_{2n},Tx_{2n})+d(\ x_{2n-1},Sx_{2n-1})\}]$$
$$+\gamma[F\{d(x_{2n},Sx_{2n-1})d(x_{2n-1},Tx_{2n})\}]$$
$$+\ \delta\ [F\{d(x_{2n},x_{2n-1})\}]$$
$$\leq\ \alpha\left[\frac{F\{d(x_{2n},x_{2n+1})d(x_{2n-1},x_{2n})d(x_{2n},x_{2n})+d(x_{2n},x_{2n-1})d(x_{2n-1},x_{2n+1})d(x_{2n},x_{2n+1})\}}{1+F\{d(x_{2n-1},x_{2n+1})d(x_{2n-1},x_{2n})\}}\right]$$
$$+\ \beta\ [F\{d(x_{2n},x_{2n+1})+d(\ x_{2n-1},x_{2n})\}]$$
$$+\ \gamma[F\{d(x_{2n},x_{2n})+d(x_{2n-1},x_{2n+1})\}]$$
$$+\delta[F\{d(x_{2n},x_{2n-1})\}]$$
$$\leq\ \alpha\left[\frac{F\{d(x_{2n},x_{2n-1})d(x_{2n-1},x_{2n+1})d(x_{2n},x_{2n+1}\ )\}}{F\{d(x_{2n-1},x_{2n+1})d(x_{2n-1},x_{2n})\}}\right]$$
$$+\ \beta\ [F\{d(x_{2n},x_{2n+1})+d(\ x_{2n-1},x_{2n})\}]+\gamma[F\{d(x_{2n-1},x_{2n+1})\}]$$
$$+\ \delta\ [F\{d(x_{2n},x_{2n-1})\}]$$
$$\leq\ \alpha\ [F\{d(x_{2n},x_{2n+1})\}]\ +\beta\ [F\{d(x_{2n},x_{2n+1})+d(\ x_{2n-1},x_{2n})\}]$$
$$+\gamma[F\{d(x_{2n-1},x_{2n})+d(\ x_{2n},x_{2n+1})\}]$$
$$+\delta\ [F\{d(x_{2n},x_{2n-1})\}]$$

$$\Rightarrow\quad F[d(x_{2n+1}x_{2n})] \leq (\beta+\gamma+\delta)\ F[d(x_{2n},x_{2n-1})]+(\ \alpha+\beta+\gamma)F[d(x_{2n},x_{2n+1})]$$
$$\Rightarrow\quad F[d(x_{2n+1}x_{2n})] \leq \left[\frac{\beta+\gamma+\delta}{1-(\alpha+\beta+\gamma)}\right] F[d(x_{2n},x_{2n-1})] \text{ where } h = \frac{\beta+\gamma+\delta}{1-(\alpha+\beta+\gamma)} < 1.$$
$$\Rightarrow\quad F[d(x_{2n+1}x_{2n})]\ \leq hF[d(x_{2n},x_{2n-1})] \leq h^2F[d(x_{2n-1},x_{2n-2})] \leq \ldots\ldots\ldots\ldots \leq h^{2n}d(\boldsymbol{x}_0,\boldsymbol{x}_1).$$

So for m > n, we have

$$\mathrm{F}[d(x_{2n},x_{2m})] \leq F[d(x_{2n}x_{2n+1})+d(x_{2n+1},x_{2n+2})+\cdots\ldots\ldots\ldots+d(x_{2m-1},x_{2m})]$$
$$\leq[h^{2n}+h^{2n+1}+h^{2n+2}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+h^{2m-1}]F[d(x_0,x_1)]$$
$$\leq\frac{h^{2n}}{1-h}F[d(x_0,x_1)]\ ,$$

we get $F\ \|d(x_{2n},x_{2m})\| \leq \frac{h^{2n}}{1-h}\ K\left[F\left\|\left(d(x_0,x_1)\right)\right\|\right]$.Hence $\lim_{nm\to\infty} d(x_{2n},x_{2m})$ =0 by (2.9),$i.e.$ $F[d(x_{2n},x_{2m})$⟶0 as n ,m⟶0.Applying lemma 1.8 (ii),$\{x_{2n}\}$ is a Cauchy sequence in X. Since (X,d) be a complete cone metric space, there exist $u \in X$ such that $\lim_{n\to\infty} x_{2n} = 0$. i.e $x_{2n} \longrightarrow \boldsymbol{u}(n \longrightarrow \infty)$.

Since $F[d(Tu,u)] \leq \alpha \left[\frac{F\{d(u,Tu)d(Tx_{2n-1},Tx_{2n})d(u,Tx_{2n}) + d(u,Tx_{2n-1})d(Tx_{2n-1},Tu)d(u,Tu)\}}{1+F\{d(Tx_{2n-1},Tu)d(Tx_{2n-1},Tx_{2n})\}}\right]$

$$
\begin{aligned}
&+\beta[F\{d(u,Tu) + d(Tx_{2n-1},Tx_{2n})\}] \\
&+\gamma[F\{d(u,Tx_{2n}) + d(Tx_{2n-1},Tu\}] \\
&+\delta\,[F\{d(u,Tx_{2n-1})\}]. \\
&\leq \alpha\left[\frac{F\{d(u,Tu)d(Tx_{2n-1},u)d(u,u) + d(u,Tx_{2n-1})d(Tx_{2n-1},Tu)d(u,Tu)\}}{1+F\{d(Tx_{2n-1},Tu)d(Tx_{2n-1},Tx_{2n})\}}\right] \\
&+\beta[F\{d(u,Tu) + d(Tx_{2n-1},u)\}]+\gamma[F\{d(u,u) + d(Tx_{2n-1},Tu\}] \\
&+\delta\,[F\{d(u,Tx_{2n-1})\}]. \\
&\leq \alpha\left[\frac{F\{d(u,Tu)\, d(u,Tx_{2n-1})d(Tx_{2n-1},Tu)\}}{F\{d(Tx_{2n-1},Tu)d(Tx_{2n-1},u)\}}\right] \\
&+\beta[F\{d(u,Tu) + d(Tx_{2n-1},u)\}]+\gamma[F\{\, d(Tx_{2n-1},Tu\}] \\
&+\delta\,[F\{d(u,Tx_{2n-1})\}] \\
&\leq \alpha\alpha[F\{d(u,Tu)\}]+\beta[F\{d(u,Tu) + d(Tx_{2n-1},u)\}] \\
&+\gamma[F\{\, d(Tx_{2n-1},Tu\}]+\delta\,[F\{d(u,Tx_{2n-1})\}]
\end{aligned}
$$

We get, $F\|d\,(Tu,u)\| \leq K\ \left[k\begin{Bmatrix}\alpha(F\|d(u,Tu)\|) + \beta(F\|d(u,Tu) + d(Tx_{2n-1},u)\|) \\ +\gamma(F\|d(Tx_{2n-1},Tu)\|) + \delta(F\|d(u,Tx_{2n-1})\|)\end{Bmatrix}\right] \longrightarrow 0.$

As $n\longrightarrow\infty$, we have $F\|d\,(Tu,u)\| = 0$ by (F_1), $\|d\,(Tu,u)\| = 0 \Rightarrow \boldsymbol{u} = Tu$ is a fixed point of T.

Now if v is another fixed point of T, then

$$
\begin{aligned}
F[d(u,v)] &\leq F[H(Tu,Tv)] \\
&\leq \alpha\left[\frac{F\{d(uTu)d(v,Tv)d(u,Tu)+d\,(u,v)d(v,Tu)d\,(u,Tu)\}}{1+d(v,Tu\,)d(v,Tv)}\right] \\
&+\beta\,[F\{d(u,Tv) + d(v,Tv)\}]+\gamma[F\,\{d(u,Tv) + d(v,Tu)\,\}]+\delta[F\{d(u,v)\}] \\
&\leq K(2\gamma+\delta)F[d(u,v)]
\end{aligned}
$$

. Hence $F\|d(u,v)\| = 0$ by (F1), $\|d(u,v)\| = 0$.

Therefore $u = v$, is a unique fixed point of T in X.

Similarly it can be established that $= u$. Hence $Tu = u = Su$. Thus $\boldsymbol{u}$ is a common fixed point of T and S.

Corollary: 4.1 Let $(X,\ d)$ be a complete cone metric space and P be a normal cone with normal constant K. Let$T,S\colon X \longrightarrow CB(X)$ be any two multivalued maps satisfying for each x, y $\in$ X,

$$
\begin{aligned}
F[H(T^m x,S^m y)] &\leq \alpha\left[F\left\{\frac{d(x,T^m x)d(y,S^m y)d(x,S^m y)+d(x,y)d(y,T^m x)d(x,T^m x)}{1+d(y,T^m x)d(y,S^m y)}\right\}\right] \\
&+\beta[F\{d(x,T^m x) + d(y,S^m y)\}] \\
&+\gamma[F\{d(x,S^m y) + d(y,T^m x)\}] \\
&+\delta\,[F\{d(x,y)\}] \qquad (2.1.17)
\end{aligned}
$$

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