Degree Equitable Domination Number and Independent Domination Number of a Graph

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Abstract: In this paper, we introduce the degree equitable domination number in graphs. Some interesting relationships are identified between domination and degree equitable domination and independent domination. It is also shown that any positive integers with \( \gamma(G) \leq \gamma_f \leq \gamma_i \) are realizable as the domination number, degree equitable domination number and independent domination number of a graph. We also investigate the degree equitable domination in the corona of graphs.

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I. INTRODUCTION

The concept of domination in graphs evolved from a chess board problem known as the Queen problem- to find the minimum number of queens needed on an 8x8 chess board such that each square is either occupied or attacked by a queen. C.Berge[3] in 1958 and 1962 and O.Ore[8] in 1962 started the formal study on the theory of dominating sets. Thereafter several studies have been dedicated in obtaining variations of the concept. The authors in [7] listed over 1200 papers related to domination in graphs in over 75 variation.

Throughout this paper, G(V, E) a finite, simple, connected and undirected graph where V denotes its vertex set and E its edge set. Unless otherwise stated the graph \( G \) has \( n \) vertices and \( m \) edges. Degree of a vertex \( v \) is denoted by \( d(v) \), the maximum degree of a graph \( G \) is denoted by \( \Delta(G) \). Let \( C_n \) a cycle on \( n \) vertices, \( P_n \) a path on \( n \) vertices and a complete graph on \( n \) vertices by \( K_n \). A graph is connected if any two vertices are connected by a path. A maximal connected subgraph of a graph \( G \) is called a component of \( G \). The number of components of \( G \) is denoted by \( \omega(G) \).

The complement \( \bar{G} \) of \( G \) is the graph with vertex set \( V \) in which two vertices are adjacent iff they are not adjacent in \( G \). A tree is a connected acyclic graph. A bipartite graph is a graph whose vertex set can be divided into two disjoint sets \( V_1 \) and another in \( V_2 \). A complete bipartite graph is a bipartite graph with partitions of order \( |V_1| = m \) and \( |V_2| = n \), is denoted by \( K_{m,n} \). A star denoted by \( K_{1,n-1} \) is a tree with one root vertex and \( n-1 \) pendant vertices. A bistar, denoted by \( B(m, n) \) is the graph obtained by joining the root vertices of the stars denoted by \( F_n \) can be constructed by identifying \( n \) copies of the cycle \( C_3 \) at a common vertex. A wheel graph denoted by \( W_n \) is a graph with \( n \) vertices formed by connecting a single vertex to all vertices of \( C_{n-1} \). A Helm graph denoted by \( H_n \) is a graph obtained from the wheel \( W_n \) by attaching a pendant vertex to each vertex in the outer cycle of \( W_n \).

The chromatic number of a graph \( G \) denoted by \( \chi(G) \) is the smallest number of colors needed to colour all the vertices of a graph \( G \) in which adjacent vertices receive different colours. For any real number \( x \), \( \lceil x \rceil \) denotes the largest integer greater than or equal to \( x \). A Nordhaus- G addum – type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. Throughout this paper, we only consider undirected graphs with no loops. The basic definitions and concepts used in this study are adopted from[11].
Given a graph $G = (V(G), E(G))$, the cardinality $|V(G)|$ of the vertex set $V(G)$ is the order of $G$ is $n$. The distance $d_G(u,v)$ between two vertices $u$ and $v$ of $G$ is the length of the shortest path joining $u$ and $v$. If $d_G(u,v) = 1$, $u$ and $v$ are said to be adjacent.

For a given vertex $v$ of a graph $G$, the open neighbourhood of $v$ in $G$ is the set $N_G(v)$ of all vertices of $G$ that are adjacent to $v$. The degree $\deg_G(v)$ of $v$ refers to $\{u \in V(G) : \{v, u\} \in E(G)\}$. The closed neighbourhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $N_G[v] = V(G)$, then $v$ is dominating in $G$. The minimum cardinality among dominating sets in $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$.

A dominating set obtained in this way above is called a degree equitable dominating set. The minimum cardinality of a degree equitable dominating set is called the degree equitable domination number of $G$.

II. RELATIONSHIPS BETWEEN DOMINATION AND INDEPENDENT DOMINATION NUMBER

A. Definition:

Given a graph $G$, choose $v_1 \in V(G)$ and put $S_1 = \{v_1\}$. If $N_G[S_1] \neq V(G)$, choose $v_2 \in V(G) \setminus N_G[S_1]$ and put if $S_2 = \{v_1, v_2\}$ and $N_G[S_2] = V(G)$, where possible, choose $v_k \in V(G) \setminus N_G[S_{k-1}]$ and put $S_k = \{v_1, v_2, \ldots, v_k\}$, there exists a positive integer $k$ such that $N_G[S_k] = V(G)$. A dominating set obtained in this way above is called a degree equitable dominating set. The minimum cardinality of a degree equitable dominating set is called the degree equitable domination number denoted by $\gamma_N(G)$.

B. Example: For all integers $n$, $\gamma_N(P_n) = \left\lceil \frac{n}{N_G(G)} \right\rceil$. Here $\gamma(G) = 2$, $\gamma_N(G) = 2$, $\gamma_G(G) = 3$. Since any independent dominating set is a degree equitable dominating set, so it follows the inequality $\gamma(G) \leq \gamma_N(G) \leq \gamma_G(G)$.

C. Example: For the spider graph, The domination number, degree equitable domination number and independent domination number follows the inequality $\gamma(G) \leq \gamma_N(G) \leq \gamma_G(G)$.

D. Result: For any regular graph $G$ with $n$ vertices is the degree equitable domination number $\gamma_N(G) = \left\lceil \frac{n}{N_G(G)} \right\rceil$.

E. Result: For any Complete graph $K_n$ is a degree equitable dominating set with $\gamma_N(K_n) = \left\lceil \frac{n}{N_G(G)} \right\rceil$.

F. Result: For any $K_n$ is a degree equitable dominating set with $\gamma_N(G) = \left\lceil \frac{n}{N_G(G)} \right\rceil$. 

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G. Result: For any unicyclic graph with even number of vertices having the pendant vertex is a degree equitable dominating set with 

\[ \gamma_N(G) = \left\lceil \frac{n}{N_G(S)} \right\rceil \]

H. Result: For any Complete Bipartite graph \( K_{n,m} \) the degree equitable dominating set with \( \gamma_N(K_{n,m}) = \left\lceil \frac{n + m}{N_G(S)} \right\rceil \)

I. Result: If \( G \) is regular graph the then \( G-e \) has a degree equitable dominating set \( \gamma_N(G-e) = \left\lceil \frac{n}{N_G(S)} \right\rceil \)

J. Result: For any wheel graph \( W_n \) has a degree equitable dominating set with \( \gamma_N(W_n) = \left\lceil \frac{n}{N_G(S)} \right\rceil \)

K. Result: If \( W_n \) is any wheel graph then \( W_n - e \) has a degree equitable dominating set with \( \gamma_N(W_n - e) = \left\lceil \frac{n}{N_G(S)} \right\rceil \)

L. Result: For any Helm graph \( H_n \) has a Hamiltonian degree equitable dominating set with \( \gamma_N(H_n) = W_n \)

III. DEGREE EQUITABLE SET OF CORONA

A. Definition:

The Corona \( G \circ H \) of a graphs \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) and \( \lvert V(G) \rvert \) copies of \( H \) and then joining the \( i^{th} \) vertex of \( G \) to every vertex in the \( i^{th} \) copy of \( H \). It is customary to denote by \( H_v \) that copy of \( H \) whose vertices are adjoined with the vertex \( v \) of \( G \). In effect \( G \circ H \) is composed of the subgraphs \( H_v + v \) joined together by the edges of \( G \). Moreover \( V(G \circ H) = \bigcup_{v \in V(G)} V(H_v + v) \)

B. Lemma: Let \( G \) and \( H \) be any graphs and let \( S \subseteq V(G \circ H) \). If \( S \) is a dominating set in \( G \circ H \) then \( S \cap V(H_v + v) \neq \emptyset \), \( \forall v \in V(G) \).

Proof: Let \( v \in V(G) \) and \( u \in V(H_v) \). If \( S \) is a dominating set in \( G \circ H \), then \( \emptyset \neq N_{G \circ H}(G \circ H) \cap S \subseteq V(H_v + v) \cap S \). This proves lemma.

In view of lemma \( \gamma_N(G \circ H) \geq \gamma_N(G) \) for any graphs \( G \) and \( H \).

C. Lemma: Let \( G \) and \( H \) be any graph and let \( S \subseteq V(G \circ H) \). If \( S \) is a degree equitable dominating set in \( G \circ H \), then \( S \cap N(H_v) \) is a degree equitable dominating set in \( H_v \).

Proof: Let \( v \in V(G) \) and \( S \subseteq V(G \circ H) \). By lemma 3.A, \( S \cap N(H_v) \neq \emptyset \). There exists \( u \in V(G) \) such that \( N_G(u) = \lvert V(S) \rvert \) then \( S_u \) is the degree equitable dominating set in \( H_v \).
D. **Theorem**: Let $G$ and $H$ be any two graphs and let $S \subseteq V(G \circ H)$. If $S$ is a degree equitable dominating set in $G \circ H$ then $S \cap N(H_v)$ is a degree equitable dominating set in $H_v$.

**Proof**: Assume $G$ is regular, To prove that $H_v$ is a degree equitable, There exists $v \in V(G)$ and $S \subseteq V(G \circ H)$. By lemma 3.B, $S_v = S \cap N(H_v) \neq \emptyset$ there exists $u \in V(G)$ such that $|N_G(u)| = |N[S_v]|$ then $S_v$ is the degree equitable dominating set in $H_v$.

Conversely, Assume that $S_v$ is degree equitable dominating set in $H_v$. To prove that $V(G)$ is regular. Suppose, $V(G)$ is not regular there exists $v \in V(G)$ and let $S_v$ By lemma 3.B, $S_v = S \cap N(H_v) \neq \emptyset$ and there exists $u \in V(G)$ such that $|N_G(u)| \neq |N[S_v]|$ which is a contradiction to our assumption that, By the definition of degree equitable dominating set $S_v$ is not a degree equitable dominating set, Hence $V(G)$ must be regular.

E. **Theorem**: Let $G$ and $H$ be any two graphs and let $S \subseteq V(G \circ H)$. If $S$ is a degree equitable dominating set in $G \circ H$, Then $S \cup N(H_v)$ is a degree equitable dominating set in $H_v$ iff one is contained in other.

**Proof**: Let $G$ and $H$ be any two graphs, Assume $S_v = S \cup N(H_v)$ is a degree equitable dominating set in $H_v$. To prove that One is contained in other. Suppose $S$ is a dominating set in $G$ and $N(H_v)$ is a set which contains every neighborhood of elements of $H_v$. Let $v \in V(G)$ and $S \subseteq V(G \circ H)$. By lemma 3.A, $S_v = S \cap N(H_v) \neq \emptyset$. By definition of $H_v$, we have $N(H_v)$ is a subset of $V(G)$ and $S \subseteq V(G)$. Therefore $N(H_v) = S$.

Conversely, Let $S = N(H_v)$, To prove that $S_v = S \cup N(H_v)$ is a degree equitable dominating set in $H_v$. There exists $v \in V(G)$ and let $S \subseteq V(G \circ H)$. By lemma 3.B, $S_v = S \cap N(H_v) \neq \emptyset$, For every $v \in V(G)$ and assumed that $S = N(H_v)$ such that $|N(G_v)| = |N(S_v)|$, Hence $S \cup N(H_v)$ is a degree equitable dominating set in $H_v$.

F. **Lemma**: Let $G$ be a regular graph with $n \geq 3$ and $H$ be any subgraph of $G$ and let $S \subseteq V(G \circ H)$. If $S$ is a dominating set in $G \circ H$, then $S$ is the induced subgraph of $G \circ H$ induced by $S$ is the Hamiltonian cycle which is a degree equitable dominating set.

**Proof**: Given $G$ be a regular graph and $H$ be any subgraph and let $S \subseteq V(G \circ H)$. By lemma 3.B. If $S$ is a dominating set in $G \circ H$, then $S \cap V(H_e + v) = \emptyset$, $v \in V(G)$. By choosing $v_1 \notin V(G \circ H)$ and put $S_1 = \{v_1\}$, if $N_G(S_1) \neq V(G \circ H)$ and $N(S_1) = H_{v_1}$, choose, $v_2 \in V(G)$ and $S_2 = \{v_1, v_2\}$ if $N_G(S_2) \neq V(G \circ H)$ and $N(S_2) = \{v_1, v_2\}$ where $k \geq 3$. Choose $v_k \notin V(G \circ H)$ and $|N_G(v_k)| = |N(G(S_k-1))|$ and put $S_k = \{v_1, v_2, ..., v_k\}$. There exists a positive integer $k$ such that $N_G(S_k) = V(G)$ where $N(S_k) = \{v_1, v_2, ..., v_k\}$, since $G$ is regular and each $N_G(S_k)$ has a $v \in G$. Hence every vertex in $S$ is adjacent to at least two vertices in $S$ as well as another vertex in $H_v$. Since $S$ is adjacent to at least two vertices in $H_v$ having a spanning cycle among the set $S$ is a Hamiltonian cycle is of equal number of degree. Hence $S$ and $H_v$ has a Hamiltonian cycle of degree equitable dominating set.

G. **Theorem**: Let $G$ be a regular graph with $n \geq 3$ and $H$ be any subgraph of $G$ and let $S \subseteq V((G - e) \circ H)$. If $S$ is a dominating set in $[(G - e) \circ H]$ then $S$ is the induced subgraph of $[(G - e) \circ H]$ induced by $S$ is the degree equitable independent dominating set in $H_v$.

**Proof**: Given $G$ be a regular graph and $H$ be any subgraph and let $S \subseteq V((G - e) \circ H)$. By lemma 3.B. If $S$ is a dominating set in $(G - e) \circ H$ then $S \cap V(H_e + v) = \emptyset$, $v \notin V(G)$. By choosing $v_1 \notin V((G - e) \circ H)$ and put $S_1 = \{H_{v_1}\}$, if $N_G(S_1) \neq V((G - e) \circ H)$ and $N(S_1) = H_{v_1}$, choose $v_2 \in V((G - e) \circ H)$ and put $S_2 = \{H_{v_1}, H_{v_2}\}$. If $N_G(S_2) \neq V((G - e) \circ H)$ and $N(S_2) = \{H_{v_2}, H_{v_2}\}$ where $k \geq 3$, choose $v_k \notin V((G - e) \circ H)$ and $|N_G(v_k)| = |N_G(S_{k-1})|$
and put $S_k = \{H_{v_1}, H_{v_2}, \ldots, H_{v_k}\}$, since $G$ is not a regular graph each $N_G(S)$ has a $H_v$. Therefore every vertex in $S$ is adjacent to only vertex in $v \in G$ but in $H_v$. Hence every vertex in $S$ is independent and degree equitable.

**H. Corrolary:**

Let $G$ be any graph with $n \geq 3$. Let $H$ be any subgraph of $G$ and let $S \subseteq V(G \circ H)$. If $S$ is a dominating set in $G \circ H$, Then $< S >$ is a induced subgraph $G \circ H$ is the degree equitable independent dominating set in $H_v$.

**Proof:** It is obvious as the previous theorem.

**REFERENCES**


