

# Finite Integrals in Association with Generalized Hypergeometric Function with Application: A Brief Review

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**ABSTRACT:** In this paper, we establish one single integral and two multiple integrals involving the product of extended Jacobi polynomials, the general class of polynomials, and  $I$ -function. We present one particular case of our first integral here that interests us. In the end, we apply our first integral by inner-connecting them with the Riemann-Liouville type of fractional integral operator.

**KEYWORDS:** Jacobi polynomials, General class of polynomials,  $I$ -function.

## I. INTRODUCTION

To unify the classical orthogonal polynomials, viz. Jacobi, Laguerre and Hermite. Fujiwara [1] defined a class of generalized polynomials using the following Rodrigues formula:

$$R_n(x) = \frac{(-1)^n k^n}{n!(x-p)^\beta (q-x)^\alpha} \frac{d^n}{dx^n} \left[ (x-p)^{\beta+n} (q-x)^{\alpha+n} \right] \quad (1.1)$$

Provided  $p < x < q$  and  $\alpha > -1, \beta > -1$ .

We denote these polynomials by  $F_n(\beta, \alpha; x)$  and call them extended Jacobi polynomials.

Thakare [2] obtained the following form of  $R_n(x) = F_n(\beta, \alpha; x)$

$$F_n(\beta, \alpha; x) = \frac{(-1)^n k^n (q-x)^n (1+\beta)_n}{n!} {}_2F_1 \left[ \begin{matrix} -n, -n-\alpha \\ 1+\beta \end{matrix}; \frac{p-x}{q-x} \right], p < x < q \quad (1.2)$$

Fujiwara [1] has proved that when  $p = 1, q = 1$  and  $k = \frac{1}{2}$

$$F_n(\beta, \alpha; x) = P_n^{(\alpha, \beta)}(x) \quad (1.3)$$

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The  $I$  -function is defined employing the following contour integrals [3]:

$$I[Z] = I_{p_i, q_i; r}^{m, n} [Z] = I_{p_i, q_i; r}^{m, n} \left[ z \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \chi(\xi) d\xi \quad (1.4)$$

where  $\omega = \sqrt{-1}$

$$\chi(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^r \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma(a_{ji}, \alpha_{ji} \xi) \right\}} \quad (1.5)$$

$p_i, q_i (i = 1, \dots, r), m, n$  are integers satisfying  $0 \leq n \leq p_i, 0 \leq m \leq q_i, (i = 1, \dots, r), r$  is finite  $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$  are real and  $a_j, b_j, a_{ji}, b_{ji}$  are complex numbers such that

$\alpha_j (b_h + v) \neq \beta_h (a_j - v - k)$  for  $v, k = 0, 1, 2, \dots$

We shall use the following notations:

$$A^* = (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i}; B^* = (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}$$

For further details of  $I$  -function, we refer to the original paper of Saxena [3]. The general class of polynomials defined by Srivastava is represented in the following manner:

$$S_\mu^\nu [x] = \sum_{k=0}^{[\mu/\nu]} \frac{(-\mu)_{\nu k}}{k!} A_{\mu, k} x^k, \quad \mu = 0, 1, 2, \dots \quad (1.6)$$

Where  $\nu$  an arbitrary positive integer, and the coefficients  $A_{\mu, \nu} (\mu, \nu \geq 0)$  are arbitrary, constant, real, or complex.

The following results will be required here:

(i) ([4], p.10, (13)), viz.

$$\int_a^b (t-a)^{x-1} (b-t)^{y-1} dt = (a-b)^{x+y-1} B(x, y); \text{Re}(x) > 0, \text{Re}(y) > 0, b < a \quad (1.7)$$

(ii) Vandermonde's theorem ([4], p.110 (4.1.2)):

$${}_2F_1(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n}; c \neq 0, 1, 2, \dots \quad (1.8)$$

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(iii) The following result:

$$(a)_n = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)} \tag{1.9}$$

## II. INTEGRALS

$$\int_p^q (x-p)^t (q-x)^\alpha F_n(\beta, \alpha; x) S_m^n [y(x-p)^u] I [z(x-p)^h] dx =$$

$$\frac{(-1)^n k^n}{n!} (q-p)^{\alpha+t+n+1} \Gamma(n+\alpha+1) \sum_{l=0}^{[n/m]} \frac{(-n)_{ml}}{l!} A_{n,l} y^l$$

$$I_{p_1+2, q_1+2; r}^{m, n+2} \left[ z(q-p)^h \left| \begin{matrix} (-t-ul, h), (\beta-t, h), A^* \\ B^*, (\beta-t+n, h), (-1-n-t-\alpha-ul, h) \end{matrix} \right. \right] \tag{2.1}$$

$$\text{Provided } \operatorname{Re} \left( t + h \frac{b_j}{\beta_j} \right) > -1, h > 0, \Omega > 0, |\arg z| < \frac{1}{2} \pi \Omega.$$

## III. SPECIAL CASES

If we put  $r = 1$  in (2.1), the  $I$ -function reduces to Fox's  $H$ -function [5], and we get

$$\int_p^q (x-p)^t (q-x)^\alpha F_n(\beta, \alpha; x) S_m^n [y(x-p)^u] H [z(x-p)^h] dx =$$

$$\frac{(-1)^n k^n}{n!} (q-p)^{\alpha+t+n+1} \Gamma(n+\alpha+1) \sum_{l=0}^{[n/m]} \frac{(-n)_{ml}}{l!} A_{n,l} y^l$$

$$H_{p+2, q+2}^{m, n+2} \left[ z(q-p)^h \left| \begin{matrix} (-t-ul, h), (\beta-t, h), A^{**} \\ B^{**}, (\beta-t+n, h), (-1-n-t-\alpha-ul, h) \end{matrix} \right. \right] \tag{3.1}$$

$$\text{Provided } \operatorname{Re} \left( t + h \frac{b_j}{\beta_j} \right) > -1, h > 0, \Delta > 0, |\arg z| < \frac{1}{2} \pi \Delta, \text{ where}$$

$$A^{**} = (a_j, \alpha_j)_{1,n}, (a_j, \alpha_j)_{n+1,p}; B^{**} = (b_j, \beta_j)_{1,m}, (b_j, \beta_j)_{m+1,q}$$

$$\Delta = \sum_{j=1}^m \beta_j + \sum_{j=1}^n \alpha_j - \sum_{j=m+1}^q \beta_j - \sum_{j=n+1}^p \alpha_j > 0$$

Because of the most general character of the result (2.1) due to the presence of an I-function, numerous special cases can be derived, but for brevity, they are not presented here.

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## IV. APPLICATION

We shall define the Riemann-Liouville fractional derivative of the function.  $f(x)$  of order  $\sigma$  (or alternatively,  $\sigma^{th}$  order fractional integral) ([6], p.181; [7], p.49) by

$${}_a D_x^\sigma \{f(x)\} = \begin{cases} \frac{1}{\Gamma(-\sigma)} \int_a^x (x-t)^{\sigma-1} f(t) dt, \text{Re}(\sigma) < 0 \\ \frac{d^q}{dx^q} {}_a D_x^{\sigma-q} \{f(x)\}, q-1 \leq \text{Re}(\sigma) < q \end{cases}, \quad (4.1)$$

Where  $q$  is a positive integer, and the integral exists. For simplicity, the special case of the fractional derivative operator  ${}_a D_x^\sigma$ , when  $a = 0$  will be written as  $D_x^\sigma$ . Thus we have

$$D_x^\sigma = {}_0 D_x^\sigma \quad (4.2)$$

Now by setting  $q = x$  in the integral (2.1), it can be rewritten as the following fractional formula:

$$\begin{aligned} & {}_p D_x^{\alpha+1} \left\{ (t-p)^\rho F_n(\beta, \alpha; t) S_m^n \left[ y(t-p)^u \right] I \left[ z(t-p)^h \right] dt \right\} = \\ & \frac{1}{\Gamma(\alpha+1)} \frac{(-1)^n k^n}{n!} (x-p)^{\alpha+t+n+1} \Gamma(n+\alpha+1) \sum_{l=0}^{[n/m]} \frac{(-n)_{ml}}{l!} A_{n,l} y^l \\ & I_{p_i+2, q_i+2; r}^{m, n+2} \left[ z(t-p)^h \Big|_{B^*, (\beta-t+n, h), (-1-n-t-\alpha-ul, h)}^{(-t-ul, h), (\beta-t, h), A^*} \right] \end{aligned} \quad (4.3)$$

Where  $\text{Re}(\alpha+1) > 0$  and all the conditions of validity mentioned in (2.1) are satisfied.

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