

Scope, Overview, Applications of Laplace Transforms in Engineering Field

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ABSTRACT: Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It finds very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing. The Laplace transform can be interpreted as a transformation from the time domain where inputs and outputs are functions of time to the frequency domain where inputs and outputs are functions of complex angular frequency.

KEYWORDS: Laplace Transform, Heaviside's, Properties, Dirac Delta, Periodic Function

I. INTRODUCTION

This paper deals with a brief overview of what Laplace Transform is and its application in the industry. The Laplace Transform is a specific type of integral transform. Considering a function $f(t)$, its corresponding Laplace Transform will be denoted as $L[f(t)]$, where L is the operator operated on the time domain function $f(t)$. The Laplace Transform of a function results in a new function of complex frequency s . Like the Fourier Transform, the Laplace Transform is also used in solving differential and integral equations. It is also predominantly used in the analysis of transient events in the electrical circuits where frequency domain analysis is used.

Definition of Laplace Transform

Consider a function of time $f(t)$, If this function satisfies Dirichlet conditions, then

$L[F(t)] = \int_0^{\infty} e^{-st} f(t) dt$ is called the laplace transformation of $f(t)$ here s can be either a real variable or a complex quantity.

II. PROPERTIES AND THEOREMS OF LAPLACE TRANSFORM

1.1 Linearity Property

If k_1 and k_2 are constants then $L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]$

1.2 Frequency shifting

If $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

1.3 Frequency differentiation

If $L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} (F(s))$

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

1.4 Frequency integration

If $L[f(t)] = F(s)$ then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s) ds$

2.1 Laplace Transform of Integral

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$

3.1 Laplace Transform of Derivatives

$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f^{(n-1)}(0) - s^{n-2} f^{(n-2)}(0) - \dots - f^{(1)}(0)$

4.1 Time shifting

$L(f(t-a)u(t-a)) = e^{-as}F(s)$ where $u(t)$ is the Heaviside step function.

5.1 Laplace Transform of Periodic Functions

Definition : A function $f(t)$ is said to be a periodic function of period $T > 0$, if $f(t + nT) = f(t)$ where $n = 1, 2, 3, \dots$

Example : $\sin(t+2n\pi) = \sin t$, $\cos(t+2n\pi) = \cos t$ are periodic functions of period 2π .

Theorem : If $f(t)$ is a periodic function of period T , then $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

Proof: We have by the definition $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$\begin{aligned} &= \int_0^\infty e^{-st} f(u) du \\ &= \int_0^T e^{-st} f(u) du + \int_T^{2T} e^{-st} f(u) du + \int_{2T}^{3T} e^{-st} f(u) du + \dots \\ &= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-st} f(u) du \end{aligned}$$

Put $u = t+nT$, $du=dt$, if $u=nT$ then $t=0$ and $u=(n+1)T$ then $t=T$

$$\begin{aligned} &= \sum_{n=0}^\infty \int_{t=0}^T e^{-s(t+nT)} f(t) dt \\ &= \sum_{n=0}^\infty e^{-snT} \int_{t=0}^T e^{-st} f(t) dt \end{aligned}$$

we've $\sum_{n=0}^\infty e^{-snT} = \frac{1}{1-e^{-sT}}$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

6.1 Unit impulse function

Definition ; The unit impulse function or Dirac delta is defined as follows

$$\delta(t - a) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t - a) ; a \geq 0$$

where $\delta_\epsilon(t - a) = \begin{cases} \frac{1}{\epsilon}, & \text{if } a \leq t < a + \epsilon \\ 0, & \text{otherwise} \end{cases}$

6.2 Laplace transform of unit impulse function:

Consider

$$L[\delta_\epsilon(t - a)] = \int_0^\infty e^{-st} \delta_\epsilon(t - a) dt = \int_0^a e^{-st} \delta_\epsilon(t - a) dt + \int_a^{a+\epsilon} e^{-st} \delta_\epsilon(t - a) dt + \int_{a+\epsilon}^\infty e^{-st} \delta_\epsilon(t - a) dt$$

i.e $L[\delta_\epsilon(t - a)] = \int_a^{a+\epsilon} e^{-st} \frac{1}{\epsilon} dt$, by using the definition

International Journal of Innovative Research in Science, Engineering and Technology

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Vol. 5, Issue 4, April 2016

$$= \frac{1}{\varepsilon} \left[\frac{e^{-st}}{-s} \right]_{\alpha}^{\alpha+\varepsilon} = \frac{-1}{\varepsilon s} \{ e^{-s(\alpha+\varepsilon)} - e^{-s\alpha} \}$$

$$\therefore L[\delta_{\varepsilon}(t-\alpha)] = e^{-s\alpha} \left(\frac{1-e^{-s\varepsilon}}{\varepsilon s} \right) \dots\dots\dots (1)$$

But $L[\delta(t-\alpha)] = L\{\lim_{\varepsilon \rightarrow 0} [\delta_{\varepsilon}(t-\alpha)]\} = \lim_{\varepsilon \rightarrow 0} L[\delta_{\varepsilon}(t-\alpha)]$

Hence taking limit on both sides of (1) as $\varepsilon \rightarrow 0$ we have,

$$\lim_{\varepsilon \rightarrow 0} L[\delta_{\varepsilon}(t-\alpha)] = e^{-s\alpha} \lim_{\varepsilon \rightarrow 0} \left\{ \frac{1-e^{-s\varepsilon}}{\varepsilon s} \right\} = e^{-s\alpha} k \dots\dots\dots(2)$$

Where $k = \lim_{\varepsilon \rightarrow 0} \left\{ \frac{1-e^{-s\varepsilon}}{\varepsilon s} \right\}$. This being in the indeterminate form $\frac{0}{0}$, we employ L'Hospital's rule to evaluate the limit.

$$\text{Hence } k = \lim_{\varepsilon \rightarrow 0} \left\{ \frac{0 - (-s)e^{-s\varepsilon}}{s} \right\} = \lim_{\varepsilon \rightarrow 0} e^{-s\varepsilon} = 1$$

Using $k = 1$ in (2) we have,

$$\lim_{\varepsilon \rightarrow 0} L[\delta_{\varepsilon}(t-\alpha)] = e^{-s\alpha}$$

Thus $L[\delta(t-\alpha)] = e^{-s\alpha}$

Note ; In particular if $a = 0$, $L[\delta(t)] = 1$

7.1 Inverse Laplace Transform

If $L[f(t)] = F(s)$ then $f(t)$ is called the inverse laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)]$

7.2 The basic table of inverse laplace transform is as follows

Sl No.	Function	Inverse Transform
1	$\frac{1}{s}$	1
2	$\frac{1}{s-a}$	e^{at}
3	$\frac{s}{s^2+a^2}$	Cos at
4	$\frac{s}{s^2-a^2}$	Cos h at
5	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
6	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
7	$\frac{1}{s^{n+1}}$ ($n > -1$)	$\frac{t^n}{\Gamma(n+1)}$
8	$\frac{1}{s^{n+1}}$ ($n > 1,2,3\dots$)	$\frac{t^n}{n!}$

7.3 Convolution

Definition: The convolution of two functions $f(t)$ and $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows

$$f(t) * g(t) = \int_{u=0}^t f(u) g(t-u) du$$

International Journal of Innovative Research in Science, Engineering and Technology

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Vol. 5, Issue 4, April 2016

7.4 Convolution theorem

Statement : If $L^{-1}[F(s)] = f(t)$, $L^{-1}[G(s)] = g(t)$ then $L^{-1}[F(s) \cdot G(s)] = \int_{u=0}^t f(u)g(t-u)du$

III. APPLICATIONS OF LAPLACE TRANSFORMS

This section describes the applications of Laplace transforms in the areas of science and engineering. The simple application in the area of Physics and Electric Circuit theory is presented below.

A Application in Physics

A very simple application of Laplace transform in the area of physics could be to find out the harmonic vibration of a beam which is supported at its two ends .Let us consider a beam of length l and uniform cross section parallel to the yz plane so that the normal deflection $w(x,t)$ is measured downward if the axis of the beam is towards x axis . The basic equation defining this phenomenon is given below.

$$EI d^4w/dx^4 - m\omega^2 w = 0 \quad (1)$$

Where E is Young's modulus of elasticity; I is the moment of inertia of the cross section with respect to the y axis ; m is the mass per unit length and ω is the angular frequency.

[1] \rightarrow by setting $\alpha^4 = \frac{m\omega^2 l^4}{EI}$

We obtain $\frac{d^4\omega}{dx^4} - \alpha^4\omega = 0 \quad (2)$

Now applying the laplace transform to (2)

we get $s^4 f(s) - s^3 f(+0) - s^2 f'(+0) - s f''(+0) - \alpha^4 f(s) = 0$
the boundary conditions of this problem are $f(+0) = 0$; $f(+l)=0$; $f'(+0)=0$; $f'(+l)=0$

Hence we obtain $f(s) = \frac{s^2 f'(+0) + f''(+0)/s^2 - \alpha^4}{s^4 - \alpha^4}$

The inverse laplace transform gives $\omega = \frac{f'(+0)}{2\alpha} \sinh \alpha x + \frac{\sin \alpha x}{\alpha} + \frac{f''(+0)}{2\alpha^3} \sinh \alpha x - \frac{\sin \alpha x}{\alpha}$

i.e $\omega = A_1 \sinh \alpha x + A_2 \sin \alpha x$

when $x = l$; $A_1 \sinh \alpha l + A_2 \sin \alpha l = 0$; $A_1 \sinh \alpha l - A_2 \sin \alpha l = 0$ these are satisfied if $A_1=A_2=0$

i.e $\sinh \alpha l = \sin \alpha l = 0$ this gives $\alpha l = n \pi$ for integral values of n Hence $A_1 = 0$ and A_2 is undetermined and the resulting vibrations are : $w_n = A_n \sin(n\pi x/l)$ and the frequencies are

$\omega_n = \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI}{m}}$ here if $n = 1$ it represents the fundamental vibration and if $n = 2$ the first harmonic and so on.

B Application in Electric Circuit Theory

The Laplace transform can be applied to solve the switching transient phenomenon in the series or parallel LR, LC or LCR circuits. A simple example of showing this application follows

Let us consider a series LCR as shown in **Figure 1**, to which a d.c. voltage V_0 is suddenly applied.

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 4, April 2016

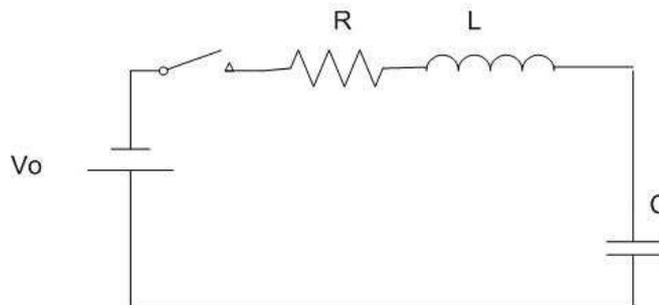


Figure 1: LCR Circuit

Now, applying Kirchhoff's Voltage Law (KVL) to the circuit, we have

$$Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt = V_0 \dots (3)$$

Differentiating both sides, $L \frac{d^2 i}{dt^2} + \frac{1}{C} i + R \frac{di}{dt} = 0$ or $\frac{d^2 i}{dt^2} + (R/L) \frac{di}{dt} + (1/LC) i = 0$

Now, applying Laplace transform to this equation, let us assume that the solution of this equation

is $i(t) = K e^{st}$ where K and s are constants which may be real, imaginary or complex.

Now, from eqn (4) $L K s^2 e^{st} + R K s e^{st} + \frac{1}{C} K e^{st} = 0$ which on simplification gives.

$$\text{or, } s^2 + (R/L) s + 1/LC = 0$$

The roots of this equation would be $s_1, s_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R^2}{4L^2}\right) - \left(\frac{1}{LC}\right)}$

The general solution of the differential equation is thus, $i(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$ where K_1 and K_2 are determined from the initial conditions

Now, if we define, $\alpha = \text{Damping Coefficient} = R/2L$ and Natural Frequency, $\omega_n = 1/\sqrt{LC}$ which is also known as undamped natural frequency or resonant frequency.

Thus, roots are; $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$

The final form solution depends on whether $(R^2/4L^2) > 1/LC$; $(R^2/4L^2) = 1/LC$;

$(R^2/4L^2) < 1/LC$; The three cases can be analysed based on the initial conditions of the circuit which are: overdamped case if $\alpha > \omega_n$, critically damped case if $\alpha = \omega_n$ and underdamped case if $\alpha < \omega_n$

IV. CONCLUSION

This paper presents a brief overview of Laplace Transform. The primary use of Laplace Transform is of converting a time domain function into its frequency domain. Major properties of Laplace Transform and a few special functions like the Heaviside's Unit Step Function and Dirac Delta Functions were discussed in detail. It also included an explanation of Inverse Laplace Transform. Laplace transform is a very effective mathematical tool to simplify very complex problems in the area of stability and control. Laplace Transform is put to tremendous use in many branches of Applied Sciences, physics and electrical power engineering.

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