

Birth of Theory of Harmonic Functions

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ABSTRACT: In this paper we define and introduce the birth of harmonic functions starting from basic definitions and theorems of harmonic starlike and convex function which is very beneficial for new researchers.

KEYWORDS: Harmonic, starlike, convex, univalent, multivalent functions, Linear operator, Hypergeometric functions

I. INTRODUCTION

The theory of Harmonic univalent and multivalent functions attracted attention of complex analysts after the appearance of a basic paper by Clunie and Sheil Small [15] in 1984. Since then the theory has rapidly developed and became an active branch of complex analysis for researchers.

G. Bernard Riemann showed that there always exists an analytic function f that maps a simply connected domain D_1 , onto a simply connected domain D_2 . This original version of the Riemann mapping theorem gave rise to the birth of geometric function theory. Koebe [25] in 1907, discovered a powerful version of the Riemann mapping theorem which state that the functions which are both analytic and univalent in the simply connected domain $D = D_1 \neq C$ maps D onto the open unit disc $\Delta = \{z : |z| < 1\}$ in such a way that $z_0 \in D$, $f(z_0) = z_0$ and $f'(z_0) > 0$. This allows pure and applied mathematicians and engineers to reduce problems about simply connected domains to the special case of the open unit disc Δ .

A univalent (m -valent) function $f(z)$ is characterized by the fact that it is analytic in the unit disc Δ and the equation $f(z) = \omega$ has never more than one (m) solution (s). A class S of univalent functions in which $f(z)$ is normalized with $f(0)=0$ and $f'(0) = 1$ is already widely studied. Failure to settle the Bieberbach conjecture until 1984 led to the introduction and investigation of several subclasses of S . An important subclass of S denoted by S^* , consists of functions that maps Δ onto a starshaped domain with respect to the origin and another important subclass of S is the family K which maps Δ onto a convex domain.

It is well known that $K \subset S^* \subset S$.

Goodman [17] introduced an interesting subclass UST of the class S^* , of starlike functions. The functions in UST are called uniformly starlike and are based on a purely geometric property. A function $f(z)$ is said to be uniformly starlike in Δ , if $f(z)$ is in S^* and has the property that for every circular arc γ contained in Δ , with centre ξ also in Δ , the arc $f(\gamma)$ is starlike with respect to $f(\xi)$. Goodman [18] also introduced an analogous subclass UCV of the class K . The function $f(z)$ is said to be uniformly convex in Δ , if $f(z)$ is in K and has the property that every circular arc γ contained in Δ with centre ξ also in Δ . The arc $f(\gamma)$ is convex. UCV denotes the class of all uniformly convex functions.

More precisely Goodman [18] proved that if $F(z) \in UST$, then there need not exist a function $f(z) \in UCV$, such that $zf'(z) = F(z)$, $z \in \Delta$, whereas Alexander's theorem asserts that a function $F(z) \in S^*$,

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if and only if there exists a function $f(z) \in K$, such that $zf'(z) = F(z)$.Rooping [29], while examining the possibility of an analogous result of Alexander's theorem for the classes UST and UCV, introduced a new class S_p of starlike functions. A function $F(z)$ is said to be in the class S_p if $F \in S^*$ and $F(z) = zf'(z)$ for some $f(z) \in UCV$. Another interesting direction of research in the theory of univalent functions was motivated by Silverman [37], when he introduced the class T of functions in S, with negative coefficients. This subclass T of S, consisting of functions analytic, univalent and with negative coefficients.

A complex-valued continuous function $f=u+iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic on D, that is u and v both satisfy the Laplace equations. In any simply connected domain D we can write $f = h + \bar{g}$, where h and g are analytic in D and h is called analytic part of f, g is called co-analytic part of f.Clunie and Sheil-Small [15] observed that a necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. A family of all harmonic complex-valued, sense-preserving univalent functions normalized with the condition $h(0) = 0 = h'(0) - 1$ is denoted by SH.

Analogous to subclasses of S, various subclasses of SH have been defined and studied.

Note that if the co-analytic part of $f = h + \bar{g}$ is identically zero i.e. $g \equiv 0$ then harmonic functions reduces to analytic functions.

II. PRELIMINARY DEFINITIONS AND THEOREMS

In this section certain preliminary definitions, basic concepts have been reviewed and certain theorems needed for the studies in this thesis have been stated, giving suitable references.

Definition 1.2.1:A function $f(z)$ defined in a domain of a complex plane is said to be univalent or schlicht in the domain, if it never takes in the domain the same value more than once, that is $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$, for all z_1, z_2 in the domain.

In this sequence, S will denote the class of all analytic and univalent functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor expansion

$$(1.2.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D.$$

Definition 1.2.2A domain D of the complex plane is said to be starlike with respect to a point $Z_0 \in D$, if for any fixed point $Z_0 \in D$ the line segment joining Z_0 to any point $Z \in D$ lies in D itself. That is, for any $Z \in D$ and $0 \leq t \leq 1$, $tZ + (1-t)Z_0 \in D$. A function $f(z) \in S$ is said to be starlike with respect to origin or simply starlike if it maps Δ onto a starlike domain with respect to origin.

Theorem 1.2.1 [28]:A necessary and sufficient condition for a function $f(z) \in S$ to be starlike univalent in Δ is that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, z \in D, S^*$$

S^* denotes the class of all starlike univalent functions.

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(An ISO 3297: 2007 Certified Organization)

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Definition 1.2.3: A domain D of the complex plane is said to be convex if the line segment joining any two points of the domain D lies entirely in the domain itself. That is if $z_1, z_2 \in D$ implies that $(tz_1 + (1-t)z_2) \in D, 0 \leq t \leq 1$. A function $f(z) \in S$ is said to be convex in Δ , if it maps Δ onto a convex domain. The subclass of functions $f(z) \in S$ which are convex univalent in the unit disc Δ is denoted by K .

Theorem 1.2.2 [28]: A necessary and sufficient condition for a function $f(z) \in S$ to be convex univalent in Δ is that

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, z \in \Delta, K \text{ denotes the class of all convex univalent functions.}$$

Definition 1.2.4 [28]: A function $f(z) \in S$ is said to be starlike of order $\alpha, 0 \leq \alpha < 1$, if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in \Delta.$$

$S^*(\alpha)$ denotes the class of starlike functions of order α . It is clear that $S^*(\alpha) \subset S$ and $S^*(0) \equiv S^*$.

Definition 1.2.5 [28]: A function $f(z) \in S$ is said to be convex of order α if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, z \in \Delta, 0 \leq \alpha < 1.$$

$K(\alpha)$ denotes the class of functions of order α .

Furthermore, $K(\alpha) \subset K$ and $K(0) \equiv K$.

Theorem 1.2.3 [17]: A function $f(z)$ given by (1.2.1) and belonging to S^* is in UST, if and only if

$$\operatorname{Re} \left\{ \frac{(z-\xi)f'(z)}{f(z)-f(\xi)} \right\} > 0, z \neq \xi$$

for every pair (z, ξ) in the poly disc $\Delta \times \Delta$.

Theorem 1.2.4 [18]: A function $f(z) \in K$ and is given by (1.2.1). Then $f(z)$ is in UCV, if and only if

$$1 + \operatorname{Re} \left\{ (z-\xi) \frac{f''(z)}{f'(z)} \right\} > 0$$

for every pair (z, ξ) in the poly disc $\Delta \times \Delta$.

Definition 1.2.6: A function $f(z)$ given by (1.2.1) is said to be in the class k -UST if and only if

$$\operatorname{Re} \left\{ (1 + ke^{i\phi}) \frac{zf'(z)}{f(z)} - ke^{i\phi} \right\} > 0$$

for $0 \leq k < \infty$ and $\phi \in \mathbb{R}$, a set of real numbers.

Definition 1.2.7: A function $f(z)$ given by (1.2.1) is said to be in the class $k - \text{UST}(\alpha)$ if and only if

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Vol. 5, Issue 7, July 2016

$$\operatorname{Re} \left\{ (1 + ke^{i\phi}) \frac{zf'(z)}{f(z)} - ke^{i\phi} \right\} > \alpha, 0 \leq \alpha < 1$$

for $0 \leq k < \infty$ and $\phi \in \mathbb{R}$, a set of real numbers.

Kanas and Wisniowska [23] studied the class of k -uniformly convex functions denoted by k -UCV which is defined as follows:

Definition 1.2.8 [23]: A function $f(z)$ given by (1.2.1) is said to be in the class k -UCV if and only if

$$\operatorname{Re} \left\{ 1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} \right\} > 0$$

for $0 \leq k < \infty$ and $\phi \in \mathbb{R}$.

Definition 1.2.9 [23]: A function $f(z)$ given by (1.2.1) is said to be in the class $k - \text{UCV}(\alpha)$ if and only if

$$\operatorname{Re} \left\{ 1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} \right\} > \alpha, 0 \leq \alpha < 1$$

for $0 \leq k < \infty$ and $\phi \in \mathbb{R}$.

Also note that

$$k - \text{UST}(0) \equiv k - \text{UST} \text{ and}$$

$$k - \text{UCV}(0) \equiv k - \text{UCV}.$$

Definition 1.2.10: A function $f(z) \in \mathcal{S}$ is said to be in the class T , if it is expressed in the form

$$(1.2.2) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, z \in \Delta.$$

This class T , introduced by Silverman [37], opened up a new direction of studies in theory of univalent functions.

Definition 1.2.11: A function $f(z) \in \mathcal{S}$ is said to be in the class T^* if $f(z)$ is starlike and is expressed in the form (1.2.2).

Salagean [33] introduced an operator, defined as follows :

For functions $f(z) \in \mathcal{S}$,

$$D^0 f(z) = f(z), D^1 f(z) = D(f(z)) = zf'(z) \text{ and}$$

$$D^v f(z) = D(D^{v-1} f(z)) = z(D^{v-1} f(z))' = z + \sum_{n=2}^{\infty} n^v a_n z^n, v \in \mathbb{N} = \{1, 2, \dots\}.$$

This operator D is called the Salagean operator.

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III. CONCLUSION

The above mentioned results and theorems based on harmonic functions are very usefully for new researchers which are new in this field as these are basics of further research.

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