ABSTRACT: A dominating set $S$ of $G$ is called a paired dominating set if $\langle S \rangle$ has a perfect matching. If $S$ is a paired dominating set with perfect matching $M$, then two vertices $u$ and $v$ are said to be paired in $S$ if the edge $uv$ is in $M$. Every graph without isolated vertices has a paired dominating set. Let $G$ be a graph without isolated vertices. The minimum cardinality of a paired dominating set of $G$ is called the paired domination number of $G$ and is denoted by $\gamma^p (G)$.

An edge $e = uv$ of a graph $G$ is said to be subdivided if $e$ is replaced by the edges $uw$ and $wv$ for some vertex $w$ not in $V(G)$. The graph obtained from $G$ by subdividing each edge of $G$ exactly once is called the subdivision graph of $G$ and is denoted by $S(G)$. In this paper, we initiate a study of the paired domination number of subdivision graphs. We determine the paired domination number of the subdivision of some standard graphs. We also obtain bounds for the paired domination number of $S(G)$.

KEYWORDS: paired domination , paired domination number, subdivision graph, Paired domination number of a subdivision graph.

INTRODUCTION

Haynes and Slater [7] introduced the concept of paired domination in graphs. Arumugam and Paulraj Joseph [1] have studied the domination parameters of the subdivision graph $S(G)$.

The graphs considered in this paper are finite, undirected, without loops, multiple edges and isolated vertices. We need the following theorems.

Theorem 1.1. For any graph $G$, $\gamma(S(G)) \leq n - 1$

Theorem 1.2. For any graph $G$, $\gamma(S(G)) \leq m(G)$

Theorem 1.3. If a graph $G$ has no isolated vertices, then $\gamma_{pr}(G) \leq 2 \gamma(G)$.

Theorem 1.4. $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2\left[\frac{n}{2}\right]$.

Theorem 1.5. If $T$ is a tree of order $n \geq 3$ with $s$ support vertices, then $\gamma_{pr}(G) \leq \frac{n + 2s - 1}{2}$.

Since $S(C_n) = C_{2n}$ and $S(P_n) = P_{2n-1}$ it follows from Theorem 1.4 that $\gamma_{pr}(S(C_n)) = 2\left[\frac{n}{2}\right]$ and $\gamma_{pr}(S(P_n)) = 2\left[\frac{2n - 1}{2}\right]$.

II. MAIN RESULTS

Theorem 2.1. For any graph $G$ of order $n$, $\gamma_{pr}(S(G)) \leq 2(n - 1)$ and this bound is sharp.

Proof: Let $G$ be any graph of order $n$. It follows from Theorem 1.1 and Theorem 1.3 that $\gamma_{pr}(S(G)) \leq 2(n-1)$. For the graph $G = K_{1,n-1}$, we have $\gamma_{pr}(S(G)) = 2(n - 1)$ and hence the inequality is sharp.

Problem 2.2. Characterize graphs $G$ for which $\gamma_{pr}(S(G)) = 2(n - 1)$.

Theorem 2.3. For any connected graph $G$, $\gamma_{pr}(S(G)) \leq 2m(G)$ where $m(G)$ is the number of edges of $G$. Further equality holds if and only if $G$ is the star $K_{1,n}$. 

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Proof. It follows from Theorem 1.2 and Theorem 1.3 that $\gamma_{pr}(S(G)) \leq 2m(G)$.

Now, suppose $\gamma_{pr}(S(G)) = 2m$. Since $\gamma_{pr}(S(G)) \leq 2(n-1)$, we have $m \leq n-1$. Since G is connected, $m \geq n-1$, so that $m = n-1$ and G is a tree. Now suppose $G \notin K_{1,n}$. Then G contains a path $P = (u_0, u_1, u_2, u_3)$.

Let $v_1$, $v_2$ and $v_3$ be the vertices of $S(G)$ subdividing the edges $u_0u_1$, $u_1u_2$ and $u_2u_3$ respectively. Let $e_i = x_iy_i$, $1 \leq i \leq n-4$ be the remaining edges of G and let $z_i$ be the vertex of $S(G)$ subdividing the edge $x_iy_i$. Now $D = \bigcup_{i=1}^{n-4} \{x_i, y_i\}$.

$\bigcup \{v_1, u_1, v_3, u_3\}$ is a paired dominating set of $S(G)$ and $|D| = 2(n-2)$. Hence $\gamma_{pr}(S(G)) \leq 2(n-2)$, which is a contradiction. Hence $G = K_{1,n}$.

Conversely, if $G = K_{1,n}$ then obviously $\gamma_{pr}(S(G)) = 2m(G)$.

Theorem 2.4. Let G be a connected unicyclic graph of order n. Then $\gamma_{pr}(S(G)) = 2(n-1)$ if and only if $G = C_2$.

Proof. Let G be a connected unicyclic graph of order n with $\gamma_{pr}(S(G)) = 2(n-1)$.

Let $C = (u_1, u_2, \ldots, u_k, u_1)$ be the cycle in G. We claim that $\deg(u_i) = 2$ for all $i = 1, 2, \ldots, k$. Suppose there exists a vertex, say $u_i$, with $\deg(u_i) \geq 3$. Then $G = [u_1u_2, u_1u_k]$ has two nontrivial components, say $G_1$ and $G_2$.

Now $\gamma_{pr}(S(G_1)) \leq 2(|V(G_1)| - 1)$ and $\gamma_{pr}(S(G_2)) \leq 2(|V(G_2)| - 1)$ so that $\gamma_{pr}(S(G)) \leq 2(n-2)$, which is a contradiction.

Thus $\deg(u_i) = 2$ for all $i = 1, 2, \ldots, k$ and $G = C$. If $G = C_n$, $n \geq 4$, then $\gamma_{pr}(S(C_n)) \leq 2\left\lfloor \frac{n}{2} \right\rfloor \leq 2(n-2)$.

Thus $n = 3$ and $G = C_3$. The converse is obvious.

Theorem 2.5. Let G be a graph of order n. Let $G^{(k)}$ be the graph obtained by attaching k pendant edges to each vertex of G. Then $\gamma_{pr}(S(G^{(k)})) = 2nk$.

Proof. Let V(G) = $\{u_1, u_2, u_3, \ldots, u_n\}$. Let $V_l = \{v_{ij}, v_{ij}, \ldots, v_{ij}\}$ be the set of pendant vertices which are adjacent to $u_i$. Let $u_{ij}$ be the vertex in $S(G^{(k)})$ which subdivides the edge $u_{ij}$.

Now $S = \{v_{ij}u_{ij} : 1 \leq i \leq k, 1 \leq j \leq k-1\} \bigcup \{u_{ij} : 1 \leq i \leq n\}$ is a paired dominating set of $S(G^{(k)})$. Hence $\gamma_{pr}(S(G^{(k)})) = 2nk$. Thus $\gamma_{pr}(S(G^{(k)})) = 2nk$.

Theorem 2.6. Let T be a tree of order n. Then $s + 1 \leq \gamma_{pr}(S(T)) \leq n + l - 1$ where l is the number of pendant vertices of T and s is the number of support vertices of T.

Proof. Let $V_l = \{u_1, u_2, \ldots, u_l\}$ be the set of all pendant vertices of T.

Let $V_2$ be the set of all pendant vertices of $G^{(k)}$ such that $|S| = 2(n(k-l) + n) = 2nk$. Further, since G has $nk$ pendant vertices, we have support vertices of T and let $V_3 = \{v_{ij}, v_{ij}, \ldots, v_{ij}\}$ where $v_i$ is the vertex subdividing the edge incident with $u_i$. Then any paired dominating set D of S(T) contains $V_2 \bigcup V_3$ or $V_1 \bigcup V_3$ and hence $\gamma_{pr}(S(T)) \geq |D| \geq s + l$.

If T is a tree with l pendant vertices, then S(T) is a tree with $(2n - l)$ vertices and $l$ supports. Now, it follows from Theorem 1.5 that $\gamma_{pr}(S(T)) \leq \frac{(2n-1 + 2l - 1)}{2} = n + l - 1$.

Thus $s + l \leq \gamma_{pr}(S(T)) \leq n + l - 1$.

Theorem 2.7. For the wheel $W_n = C_{n-1} + K_1$, we have $\gamma_{pr}(S(W_n)) = 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. Let V($W_n$) = $\{u_0, u_1, u_2, \ldots, u_{n-1}\}$, and

$E(W_n) = \{u_iu_{i+1} : 1 \leq i \leq n-2\} \bigcup \{u_iu_{i-1}\} \bigcup \{u_iu_l : 1 \leq i \leq n-1\}$

Let $v_i$ be the vertex subdividing the edge $u_iu_{i+1}$, $1 \leq i \leq n-2$ and let $w_{n-1}$ be the vertex subdividing the edge $u_{n-1}u_0$. Let $D$ be a minimum paired dominating set of the cycle
Hence dominates pair \((W_1, W_2)\), so that \(|D| = 2\left\lfloor \frac{n-1}{2} \right\rfloor\). Now, \(D = D_n \cup \{u_0, v_1\}\) is a paired dominating set of \(S(W_n)\) and hence \(\gamma_{pr} (S(W_n)) \leq 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor\).

Now, let \(D\) be any paired dominating set of \(S(W_n)\).

**Case i.** \(u_0 \notin D\).

Since \(D\) is a paired dominating set, we may assume without loss of generality that \(v_1, u_1 \in D\). Now, for each \(i, 2 \leq i \leq n-1\), at least one of the vertices \(u_i, v_i\) is in \(D\) and since \(D\) is a paired dominating set, it follows that \(|D| \geq 2n - 2\).

**Case ii.** \(u_0 \in D\).

Since \(D\) is a paired dominating set, we may assume without loss of generality that \(v_1 \in D\). Now, \(D - \{u_0, v_1\}\) is a paired dominating set of the path \((W_1, W_2, W_3, \ldots, W_{n-1}, W_n)\). Hence it follows that, \(|D| \geq 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor\).

\[
\gamma_{pr} (S(W_n)) \geq 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor. \text{ Thus } \gamma_{pr} (S(W_n)) = 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor.
\]

**Theorem 2.8.** For the complete bipartite graph \(K_{m,n}\) with \(m \leq n\), we have \(\gamma_{pr} (S(K_{m,n})) = 2n\).

**Proof.** Let \(V_1 = \{u_1, u_2, \ldots, u_m\}\) and \(V_2 = \{v_1, v_2, \ldots, v_n\}\) be a bipartition of \(K_{m,n}\).

Let \(n = m + r\) where \(r \geq 0\). Let \(w_i (1 \leq i \leq m, 1 \leq j \leq n)\) be the vertex of \(S(K_{m,n})\) which is adjacent to \(u_i\) and \(v_j\).

Let \(S = \bigcup_{j=1}^{n} \{v_j, w_j\} \bigcup \bigcup_{j=m+1}^{n} \{v_j, w_j\}\). Clearly \(S\) is a paired dominating set of \(S(K_{m,n})\) and hence \(\gamma_{pr} (S(K_{m,n})) \leq 2n\).

Also, any paired dominating set of \(S(K_{m,n})\) contains at least \(n\) vertices from \(K_{m,n}\) and at least \(n\) vertices from \(V (S(K_{m,n})) - V (K_{m,n})\) so that \(\gamma_{pr} (S(K_{m,n})) \geq 2n\). Thus \(\gamma_{pr} (S(K_{m,n})) = 2n\).

**Theorem 2.9.** For the complete graph \(K_n\) we have \(\gamma_{pr} (S(K_n)) = 2(n-1)\).

**Proof.** It follows from Theorem 2.1 that \(\gamma_{pr} (S(K_n)) \leq 2(n-1)\). Now, let \(V (K_n) = \{v_1, v_2, \ldots, v_n\}\) and let \(W = V (S(K_n)) - V (K_n) = \{w_1, w_2, \ldots, w_n\}\) where \(r = \left\lfloor \frac{n}{2} \right\rfloor\).

Let \(S\) be any paired dominating set of \(S(K_n)\) with \(|S| = 2k, k \geq 1\). Then \(S\) contains \(k\) vertices of \(V (K_n)\) and \(k\) vertices of \(W = V (S(K_n)) - V (K_n)\) we may assume without loss of generality that \(S = \{v_1, w_1, v_2, w_2, \ldots, v_{k}, w_{k}\}\) where \(k \leq n\).

Consider the first pair \((v_1, w_1)\) of vertices in \(S\). This pair of vertices dominate \((n-1)\) vertices in \(W\). Now, the second pair \((v_2, w_2)\) dominates \((n-2)\) vertices in \(W\) other than the vertices dominated by \((v_1, w_1)\). In general the pair \((v_i, w_i)\) dominates \(n-i\) vertices of \(W\), other than the vertices dominated by \((v_1, w_1), \ldots, (v_{i-1}, w_{i-1})\). Since \(S\) dominates all the \(n\) vertices of \(W\), it follows that \(\frac{n(n-1)}{2} = nk - \frac{k(k-1)}{2}\). Hence \(k^2 + k(1-n) + (n^2 - n) = 0\). Solving this equation for \(k\), we get \(k = n\) or \(k = n - 1\).

Hence \(|S| \geq 2(n-1)\) so that \(\gamma_{pr} (S(K_n)) \geq 2(n-1)\). Thus \(\gamma_{pr} (S(K_n)) = 2(n-1)\).

**III. CONCLUSION**

We have found paired domination number of the subdivision of some standard graphs. Using the proof technique we can find paired domination number of the subdivision of some other graphs also.
REFERENCES