

# A Novel Explicit Closed Form Solution to the Nonlinear Buckling Loads of a Graphene Beam under Axial Compressive Load

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**ABSTRACT:** In this paper, we introduce analytical closed form solution to the buckling loads of Euler graphene beam. The buckling load is expressed as a function of the graphene nonlinearity and buckling mode shapes. The problem of nonlinear buckling of a graphene beam was analytically investigated using the averaging method and an approximate solution was obtained. In this solution, a transcendental cubic equation including the buckling load and graphene nonlinearity parameter could be solved numerically for the buckling load. In the current paper, this equation is solved analytically for the buckling load in a closed form. The buckling loads and deformation at any point along the beam are explicitly expressed as functions of graphene nonlinearity and mode shapes. Therefore, a parametric study can be pursued to predict the behaviour of the buckling load at higher nonlinearity. The new solution showed the same behaviour as in the previously obtained numerical ones. However, a parametric study can now be pursued to show the effects of mode shapes and nonlinearity on the buckling loads analytically and graphically. There are also some other previous analytical and numerical solutions which start with the assumptions of certain mode shape, the current solution is valid for any mode shape.

**KEYWORDS:** Euler-Bernoulli beam, graphene material, buckling loads, averaging method.

## I. INTRODUCTION

The buckling problem of a simply supported beam, made of linear material, under axial compressive load is already solved analytically for buckling load and deflection at any point along the beam. The buckling load depends on the length of the beam and the buckling mode shape. The deflection has a form of a sine function with constant amplitude and phase angle. On the other hand, the graphene material assumes nonlinear stress strain relation. The nonlinearity is expected to introduce new parameters in the solution of the buckling load and deflection of a simply supported beam under axial compressive load made of graphene.

The problem of buckling of graphene beam was treated analytically using the perturbation of eigen value problem [1]. The perturbation solution allows obtaining the buckling load for the first mode shape only. Same problem was treated numerically using the shooting method and finite difference method [1,2]. In a previous work, the problem of nonlinear buckling of a graphene beam, simply supported, under compressive axial load was investigated analytically [3]. When assuming that the amplitude of deflection shape is slowly varying along the beam, the averaging method can be applied to obtain the change in the amplitude and the phase angle at any point on the beam. An algebraic transcendental equation includes the buckling load, the graphene nonlinearity, the mode shape, and the amplitude was derived. This equation could be solved numerically for the buckling loads with graphene nonlinearity at certain mode shapes.

In this paper, the Euler equation of the graphene beam will be solved analytically providing a closed form solution to the buckling loads. In the second section the equation of equilibrium of the Euler graphene beam will be reviewed and necessary equations will be introduced. In the third section, the closed form solution to the buckling load as a function of graphene nonlinearity and buckling mode shapes is derived. In the fourth section, results are discussed and compared with solutions from previous researches. Then, a parametric study is performed analytically and represented graphically

to show the effect of graphene nonlinearity on buckling loads. In the fifth section, the mathematical significance and properties of the solution are analyzed. Finally, in the sixth section conclusions are drawn.

## II. GRAPHENE BEAM'S EQUATION

Figure 1 shows a simply supported graphene beam, where  $L$  is its length, subject to a compressive axial load  $P$ . The problem of buckling of this beam is well established when the stress strain relation follows the Hook's law. However, in case of graphene the stress strain relation is nonlinear and takes the following form [4-6]

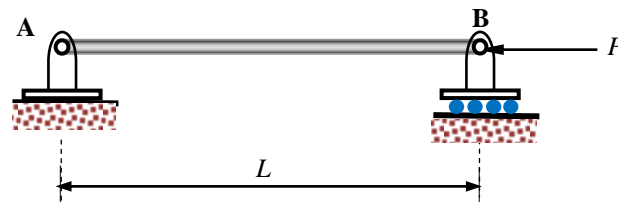


Fig. 1. Simply Supported Beam

$$\sigma = E\varepsilon_x + D|\varepsilon_x|\varepsilon_x \quad (1)$$

where  $E$  is the Young's modulus of the graphene and  $D = -E^2 / 4\sigma_{\max}^2$  is a nonlinear third order elastic stiffness. For graphene, it is found that  $E = 340 \pm 40 \text{ N/m}$  and  $D = -690 \pm 120 \text{ N/m}$  [7,8]. A normalized form of the equation of equilibrium of this beam can be formulated as [1-3]

$$u'' - \alpha(|u''|u'') + \lambda u = 0 \quad (2)$$

where

$$\alpha = \frac{PL^2}{EI_2}, \quad \lambda = \frac{|D|I_3}{EI_2L} \quad (3)$$

with the homogeneous Dirichlet boundary conditions

$$u(0) = u(1) = 0 \quad (4)$$

where  $u$  is the deflection at any point along the beam,  $\alpha$  is a constant represents the nonlinearity of the graphene material, and  $\lambda$  represents the buckling load. For full derivation of (1) the reader is referred to [1-3]. When neglecting the nonlinearity and setting  $\alpha = 0$  the problem simplifies to the regular Euler-Benoulli beam equation with the deflection takes the form of a circular function

$$u(x) = A \sin(\omega x + \phi) \quad (5)$$

where the amplitude  $A$  and the phase angle  $\phi$  are constants. The averaging method was used to predict the effect of nonlinearity of the graphene beam on its amplitude and phase [3,9]. When substituting (5) into (1) and considering  $A$  and  $\phi$  functions of time then separate the equation of the derivative of  $A$  and  $\phi$  respectively one has

$$\frac{dA}{dx} = -\frac{\alpha A^2 \omega^3}{2\pi} \int_0^{2\pi} |\sin(\psi)| \sin(\psi) \cos(\psi) d\psi = 0 \quad (6)$$

$$\frac{d\phi}{dx} = \frac{\alpha A^2 \omega^3}{2\pi} \int_0^{2\pi} |\sin(\psi)| \sin^2(\psi) d\psi = \frac{4\alpha A \omega^3}{3\pi} \quad (7)$$

Where  $\psi = (\omega x + \phi)$ . Eq. (6) shows that the average change of the amplitude along the beam's axis is zero, i.e. the nonlinearity will maintain the amplitude on the average, while (7) shows that the phase is linearly changing along the axis of the beam. Then, the phase angle can take the form

$$\phi(x) = \frac{4\alpha A\omega^3}{3\pi} x + \phi_0 \tag{8}$$

Using (8) to rewrite the solution in (5) gives

$$u(x) = A \sin \left( \omega \left( 1 + \frac{4\alpha A\omega^2}{3\pi} \right) x + \phi_0 \right) \tag{9}$$

It is obvious from (9) that the nonlinearity manifests itself into the fact that the frequency of mode shapes becomes a function of the amplitude and graphene nonlinearity. It is obvious that if  $\alpha = 0$ , (9) simplifies to the solution of the linear problem (5). To satisfy the boundary conditions in (4) the following transcendental equation is obtained.

$$\frac{4\alpha A}{3\pi} \omega^3 + \omega - n\pi = 0 \tag{10}$$

Equation (10) can be solved numerically to obtain  $\omega$ , and then  $\lambda$ , with  $\alpha$  at different values of the mode shape  $n$  [3]. This was the best result obtained to find the buckling load of a graphene beam among other previous trials, either analytically using the perturbation method or numerically using the finite difference method [1,2]. Perturbation solution based on expansion in terms of a small parameter has convergence problems at larger values of  $\alpha$ . For example, solution using eigen value perturbation method proves to be valid for  $\alpha$  ranges from 0 to 0.1 [1]. Numerical solution using finite difference method applies only for the first mode where the deflection is positive at any point on the beam [2].

### III. CLOSED FORM SOLUTION OF BUCKLING LOAD

After publishing [3] the idea came to us to solve (10) analytically for  $\omega$ . It is interesting to know if  $\omega$  can be obtained as a function of  $\alpha, n$ , and  $A$  i.e.  $\omega = \omega(\alpha; n; A)$ . In addition, if such function exists what are the characteristics of it? Equation (10) can be written as

$$\omega^3 + \beta\omega - \gamma = 0, \tag{11}$$

Where,  $\beta = (3\pi / 4\alpha A)$  and  $\gamma = n\pi\beta$ . Substituting  $\omega = \sqrt{4\beta/3} \sinh \theta$  in (11), where  $\theta$  is a dummy variable, and after some algebra we obtain the following exact form for  $\omega$

$$\omega = \frac{(3n\pi/2)}{\delta} \left\{ \left( \delta + \sqrt{\delta^2 + 1} \right)^{\frac{1}{3}} - \left( \delta - \sqrt{\delta^2 + 1} \right)^{\frac{1}{3}} \right\} \tag{12}$$

where

$$\delta = 3n\sqrt{\pi\alpha A} \tag{13}$$

Knowing that  $\lambda = \omega^2$  one obtains

$$\lambda = \left( \frac{3n\pi}{2} \right)^2 \left\{ \left( \frac{\left( \delta + \sqrt{\delta^2 + 1} \right)^{\frac{1}{3}}}{\delta} \right)^2 + \left( \frac{\left( \delta - \sqrt{\delta^2 + 1} \right)^{\frac{1}{3}}}{\delta} \right)^2 - \frac{2}{\delta^2} \right\} \tag{14}$$

Comparing the two equations (12) and (14) one finds that the expression of  $\lambda$  in (14) is cumbersome. Thus, one can choose to proceed in further analysis using (12). One way to check the solution in (12) is to prove that it is reduced to the linear case when the nonlinearity vanishes i.e.  $\lim_{\alpha \rightarrow 0} \omega = n\pi$ . From (13), this is equivalent to prove that

$\lim_{\delta \rightarrow 0} \omega = n\pi$ . Rewriting(12) on the form

$$\omega = \frac{3n\pi}{2} F(\delta) \tag{15}$$

where

$$f(\delta) = \frac{\left(\delta + \sqrt{\delta^2 + 1}\right)^{\frac{1}{3}} - \left(\delta - \sqrt{\delta^2 + 1}\right)^{\frac{1}{3}}}{\delta} \tag{16}$$

Using L'Hôpital's rule we can prove that

$$\lim_{\delta \rightarrow 0} F(\delta) = \frac{2}{3} \tag{17}$$

and as a result  $\lim_{\delta \rightarrow 0} \omega = n\pi$ . It is also interesting to note that, there is no real positive value of  $\delta$  satisfies the equation  $df(\delta)/d\delta = 0$ . Thus,  $\omega$  has no extremum and monotonically increases or decreases. The fact that the function  $f(\delta)$  has no extremum point makes it difficult to express  $\omega$  as series expansion in  $\delta$ . As there is no region in which  $f(\delta)$  can be considered analytic in  $\delta$  or  $\alpha$ . From this fact, we conclude that solutions based on series expansion of buckling load in  $\alpha$  will have convergence problem. Also, we can justify why previous researches used same series expansion method couldn't continue to higher values of  $\alpha$ .

#### IV. RESULTS AND DISCUSSION

Figure 2 shows a 3D representation of  $\lambda$  with  $\alpha$ , and  $n$ . It is shown that  $\lambda$  decreases with increasing  $\alpha$  and increases with increasing  $n$ . Figure 3 shows contour representation of  $\lambda$  with  $\alpha$ , and  $n$  so that each contour line represents one buckling load. The contour line represents  $\lambda = 10$  can excite the mode  $n = 3$  at  $\alpha = 0.46707$  and the mode  $n = 4$  at  $\alpha = 0.7008$ . It is interesting to note that to excite the mode  $n = 4$  at  $\alpha = 1.54$  requires buckling load  $\lambda = 5$ .

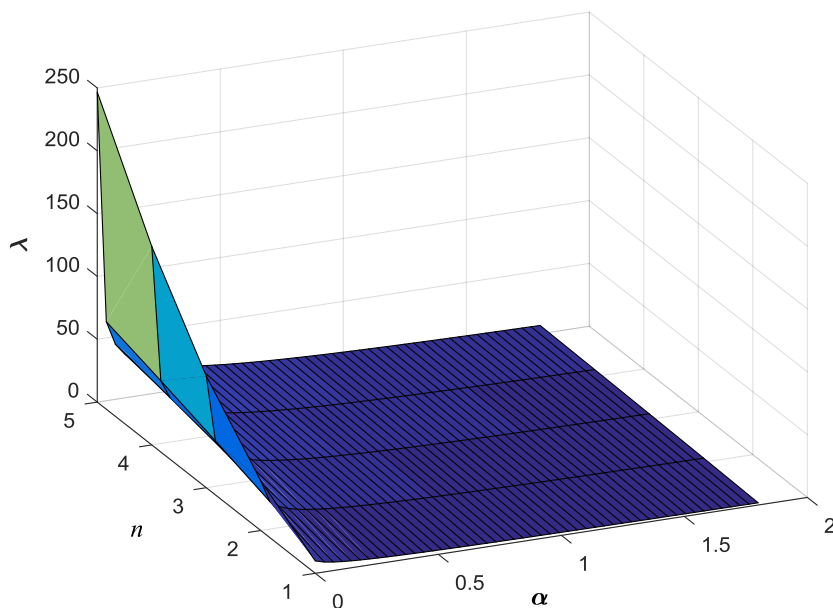


Fig. 2. 3D plot of values of  $\lambda$  with  $\alpha$  and  $n$

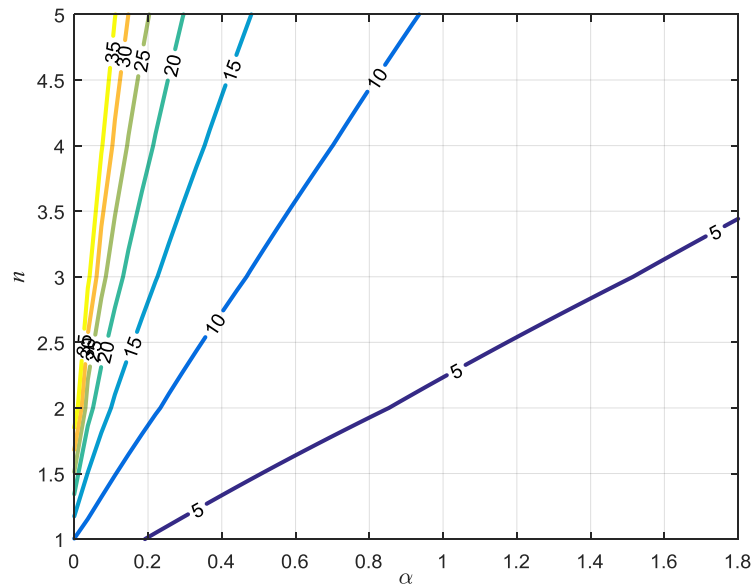


Fig. 3. Contour values of  $\lambda$  with  $\alpha$  and  $n$

V. MATHEMATICAL SIGNIFICANCE OF THE SOLUTION

The obtained solution looks interesting from mathematical point of view. Equation (16) can be written on the general form

$$F(\delta; \varepsilon) = \frac{(f(\delta))^\varepsilon - (f(\delta))^{-\varepsilon}}{\delta} \tag{18}$$

The mathematical form of  $F(\delta; \varepsilon)$  is close to some class of hyperbolic functions Specially, Fibonacci and Lucas functions [10,11].

The condition on  $f(0) = 1$  should be satisfied. Also, there is no value of  $\delta$  that satisfies the equation  $f(\delta) = 0$ .

When used in certain application, more conditions on  $f(\delta)$  and  $\frac{df(\delta)}{d\delta}$  result in determining appropriate value of  $\varepsilon$ .

In our special case

$$\varepsilon = \frac{1}{3}, f(\delta) = \delta + \sqrt{\delta^2 + 1} \tag{19}$$

and figure 4 represents the current function  $f(\delta)$  for  $\delta \geq 0$ . It is obvious that  $f(\delta)$  is monotonically increases with  $\delta$ . At larger values of  $\delta$ , this relation can be represented by a straight line with a constant slope of  $\frac{df(\delta)}{d\delta} = 2$ .

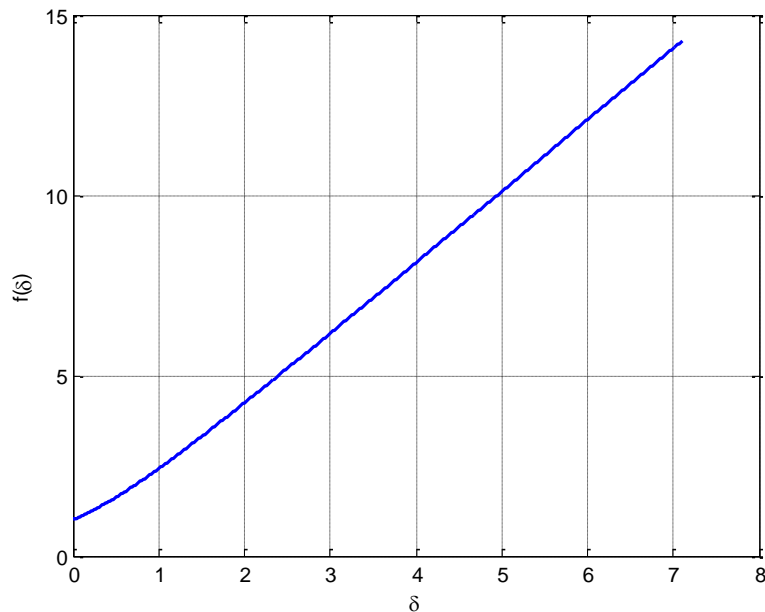


Fig. 4. The function  $f(\delta)$  for  $\delta \geq 0$

In our solution the argument  $\delta$  is not the actual independent variable. It is a function of a physical variable  $\alpha$  as indicated by (13). It is interesting to see the relation between  $f(\delta)$  and  $\alpha$  which is indicated in figure 5. In this figure, the slope decreases with  $\alpha$  but still at higher values of  $\alpha$  the relation can be approximated by a straight line.

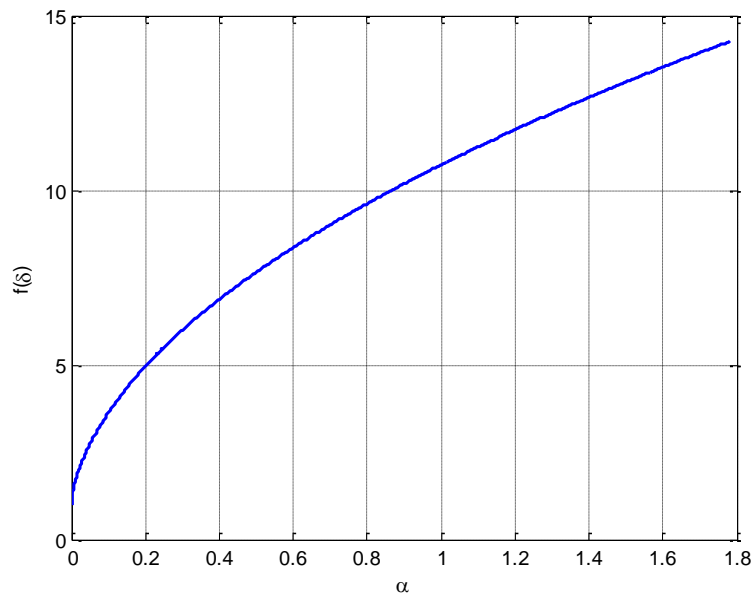


Fig. 5. The function  $f(\delta)$  for  $\alpha \geq 0$

Figure 6 shows the function  $F(\delta; \varepsilon)$  with  $\alpha \geq 0$  when  $\varepsilon = 1/3$ . It starts at  $F(0; \frac{1}{3}) = \frac{2}{3}$  and then monotonically decreases. From equation (15), this behavior means that the buckling loads decrease with increasing nonlinearity. This decrease is sharp at the beginning then becomes smoother at high nonlinearity. Similarly one can generate more graphs

when including different values of  $\varepsilon$ . A three dimensional graph shows how  $F(\delta; \varepsilon)$  changes with both  $\delta$  and  $\varepsilon$  will be interesting. But, as mentioned above the specific value of  $\varepsilon$  as a parameter depends on the specific mathematical application in which the solution is implemented.

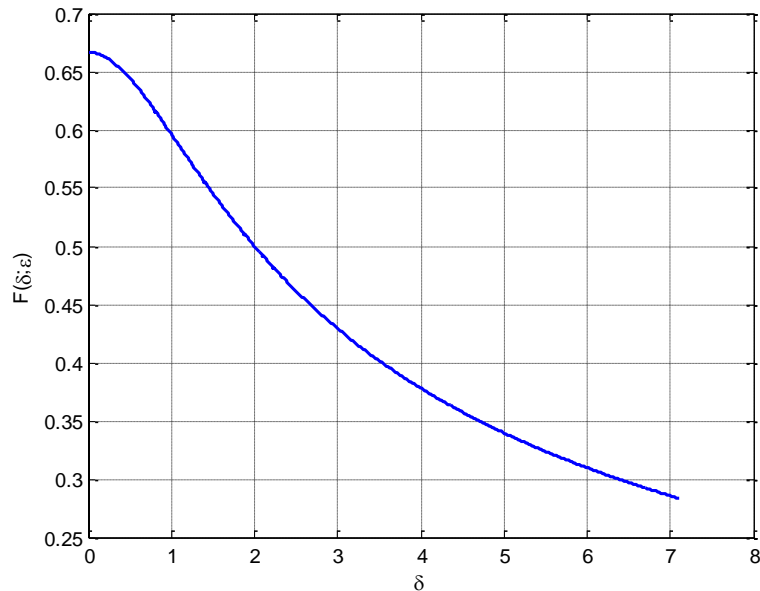


Fig.6. The function  $F(\delta; \frac{1}{3})$  for  $\delta \geq 0$

If we fix the form of  $f(\delta)$  and change  $\varepsilon$ , other class of  $F(\delta; \varepsilon)$  functions can be obtained. It will be interesting in a future research to investigate the properties of this class of functions and its potential applications.

## VI. CONCLUSION

A closed form analytical solution to the buckling load of graphene beam could be derived. As a result, a full parametric study including the effect of nonlinearity and mode shapes on buckling load are analyzed analytically then represented graphically. The obtained solution shows some interesting and unique mathematical properties that can be the subject of a future research.

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