

The Product of increasing and bounded function in Laplace Integral Transform

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ABSTRACT : In this paper the Product of increasing and bounded function in Laplace Integral Transform. The fourier transform tells about a function or signal as a series of modes of vibrations, where as the Laplace transform tells about a function into its moments. The Laplace transform is used for solve the differential equation and integral equations. In physics and engineering the Laplace transform is used for analysis of linear time invariant system. In the kind of analysis, the Laplace transform is often assumed as a transform from the time domain, in which inputs and outputs are functions of complex angular frequency.

KEYWORDS : Laplace transform, unilateral transforms, Bounded function, Monotonic functions, Laplace–Stieltjes transform, Fourier transform.

I INTRODUCTION

The method of Laplace Transform has the advantage of directly giving the solution of differential equation with boundary values without. The necessary of first finding the general solution and then evaluating from it the arbitrary constant moreover the ready tables of Laplace Transforms reduce the problem of solving differential equation to more algebraic manipulation[4].

The Laplace transform is also related to the fourier transform, but the fourier transform tells about a function or signal as a series of modes of vibrations, where as the Laplace transform tells about a function into its moments[12]. Like the fourier transform, the Laplace transform is used for solve the differential equation and integral equations. In radian per unit time. It is given that mechanical or functional description of an input or output to a system. Also the Laplace transform provides an alternative functional description that after simplify the process of analyzing the behavior of the system, or in making a new system based on a set of specifications.

II RELATED WORK

2.1 Integral transform are applied and helpful in the solution of partial differential equation. But, the choice of a particular transform to be used for the solution of a differential equation depends on the characteristic of the boundary condition of the equation[7-9].

And the facility with which the transform $f(p)$ can be inverted to provide -

$$f(s) = \int_{-\infty}^{\infty} e^{-st} d\alpha(t) \tag{2.1,1}$$

Here, Let $\alpha(t)$ be of closely related variation in every finite interval. Particularly, if $\alpha(t)$ is an integral of a function $\phi(t)$, the integral (2.1,1) becomes,

$$f(s) = \int_{-\infty}^{\infty} e^{-st} \phi(t) dt \tag{2.1,2}$$

Laplace transform or two sides Laplace transform by extending the limit of integration to be the entire real excess if that is done the common unilateral transform simply becomes a special case of integral transform

We say that $\alpha(t)$ is normalized in $(-\infty, \infty)$ if and only if $\alpha(0) = 0$ and

$$\alpha(t) = \frac{\alpha(t+) + \alpha(t-)}{2} \quad (-\infty < t < \infty). \tag{2.1,3}$$



We observed a function normalized in interval $(0, \infty)$ and zero in $(-\infty, 0)$ is not necessarily normalized in $(-\infty, \infty)$.

For the given values of s, whenever the integral (1.1,1) converges;

We found-

$$f(s) = \int_0^\infty e^{-st} d\alpha(t) + \int_0^\infty e^{st} d[-\alpha(-t)] \tag{2.1,4}$$

So, the study of the bilateral transform is lowered to the total of two unilateral transforms in one of which the variable s has been substituted by -s.

III MATERIALS AND METHODS

3.1 Fourier transform

The continuous Fourier transform is equivalent to evaluating the bilateral Laplace transform with imaginary argument $s = i\omega$ or $s = 2\pi fi$:

$$\begin{aligned} \hat{f}(\omega) &= F\{f(t)\} \\ &= L\{f(t)\}\Big|_{s=i\omega} = F(s)\Big|_{s=i\omega} = \int_{-\infty}^\infty e^{-i\omega t} f(t) dt. \end{aligned}$$

This expression excludes the scaling factor $1/\sqrt{2\pi}$, which is often included in definitions of the Fourier transform. This relationship between the Laplace and Fourier transforms is often used to determine the frequency spectrum of a signal or dynamical system[12-14]. The above relation is valid as stated if and only if the region of convergence (ROC) of $F(s)$ contains the imaginary axis, $\sigma = 0$. For example, the function $f(t) = \cos(\omega_0 t)$ has a Laplace transform $F(s) = s/(s^2 + \omega_0^2)$ whose ROC is $\text{Re}(s) > 0$. As $s = i\omega$ is a pole of $F(s)$, substituting $s = i\omega$ in $F(s)$ does not yield the Fourier transform of $f(t)u(t)$, which is proportional to the Dirac delta-function $\delta(\omega - \omega_0)$.

However, a relation of the form

$$\lim_{\sigma \rightarrow 0^+} F(\sigma + i\omega) = \hat{f}(\omega) \tag{1.5,9}$$

holds under much weaker conditions. For instance, this holds for the above example provided that the limit is understood as a weak limit of measures (see vague topology). General conditions relating the limit of the Laplace transform of a function on the boundary to the Fourier transform take the form of Paley-Wiener theorems.

3.2 Laplace–Stieltjes transform

The (unilateral) Laplace–Stieltjes transform of a function $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by the Lebesgue–Stieltjes integral[2].

$$\{L^*g\}(s) = \int_0^\infty e^{-st} dg(t).$$

The function g is assumed to be of bounded variation. If g is the antiderivative of f :

$$g(x) = \int_0^x f(t) dt$$

then the Laplace–Stieltjes transform of g and the Laplace transform of f coincide. In general, the Laplace–Stieltjes transform is the Laplace transform of the Stieltjes measure associated to g . So in practice, the only distinction between the two transforms is that the Laplace transform is thought of as operating on the density function of the measure, whereas the Laplace–Stieltjes transform is thought of as operating on its cumulative distribution function.

3.3 Bilateral Laplace transform

When one says "the Laplace transform" without qualification, the unilateral or one-sided transform is normally intended. The Laplace transform can be alternatively defined as the bilateral Laplace transform or two-sided Laplace transform by extending the limits of integration to be the entire real axis. If that is done the



common unilateral transform simply becomes a special case of the bilateral transform where the definition of the function being transformed is multiplied by the Heaviside step function[12].

The bilateral Laplace transform is defined as follows:

$$F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt.$$

IV RESULTS AND DISCUSSION

4.1 The Product of increasing and bounded function

Let $\alpha(t)$ be to in $(-\infty, \infty)$. Hence, by Fourier-Stieltjes transform :-

$$F(x) = \int_{-\infty}^{\infty} e^{ixt} d\alpha(t),$$

and this integral actually converges for all real x. We hope to determine the product of two such function F(x). We require-

Axiom 4.1(a) If for each non-negative number R the function $\psi_R(t)$ is increasing and bounded, so if;

$$\lim_{R \rightarrow \infty} \psi_R(t) = \gamma(t) \quad (-\infty < t < \infty) \quad (4.1,1)$$

$$\lim_{R \rightarrow \infty} \psi_R(\pm \infty) = \gamma(\pm \infty), \quad (4.1,2)$$

Then-

$$\lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} e^{ixt} d\psi_R(t) = \int_{-\infty}^{\infty} e^{ixt} d\gamma(t) \quad (41,3)$$

Clearly $\gamma(t)$ is increasing and bounded as a result, the integral on the right-hand side of (4.1,3) remains. Set

$$I_R = \int_{-\infty}^{\infty} e^{ixt} d[\psi_R(t) - \gamma(t)],$$

Evaluating it as the total of three integrals I'_R, I''_R, I'''_R Similarly, to the three intervals of integration $(-\infty, -A), (-A, A)$ and (A, ∞) , simultaneously where A is an arbitrary non-negative constant. Then;

$$|I'_R| \leq \int_{-\infty}^{-A} d[\psi_R(t) + \gamma(t)] = \psi_R(-A) + \gamma(-A) - \psi_R(-\infty) - \gamma(-\infty)$$

$$|I'''_R| \leq \int_A^{\infty} d[\psi_R(t) + \gamma(t)] = \psi_R(\infty) - \psi_R(A) - \gamma(A)$$

Hence,

$$|I_R| \leq \left| \int_{-A}^A e^{ixt} d[\psi_R(t) - \gamma(t)] \right| + \psi_R(-A) + \gamma(-A) - \psi_R(-\infty) - \gamma(-\infty) + \psi_R(\infty) + \gamma(\infty) - \psi_R(A) - \gamma(A)$$

$$\overline{\lim}_{R \rightarrow \infty} |I_R| \leq 2\gamma(-A) - 2\gamma(-\infty) + 2\gamma(\infty) - 2\gamma(A).$$

Here, the right-hand side of this inequality can be made arbitrarily small as per the choice of A, & the left-hand side is not dependent of A. Consequently the latter is the number zero, thus; proved.



Using, lemma proof can be done of the fundamental theorem regarding the product of increasing and bounded function.

Axiom 4.1(b) If $\alpha(t)$ and $\beta(t)$ are of closely-related & followed in $(-\infty, \infty)$ and if $\gamma(t)$ is their Stieltjes outcome, then-

$$\int_{-\infty}^{\infty} e^{ixt} d\alpha(t) \int_{-\infty}^{\infty} e^{ixt} d\beta(t) = \int_{-\infty}^{\infty} e^{ixt} d\gamma(t) \quad (-\infty < x < \infty) \quad (4.1,4)$$

It will be adequate to take the case in which $\alpha(-\infty) = \beta(-\infty) = 0$. On subtraction a constant from $\alpha(t)$ and one from $\beta(t)$. we only alters $\gamma(t)$ by the addition of a constant in order to make the equation (4.1,4) as it is. Also by decomposition of $\alpha(t)$ and $\beta(t)$ first into actual & imaginary parts and then into monotonic functions we observe that there is hardly any limitation in assuming that $\alpha(t)$ and $\beta(t)$, and hence $\gamma(t)$, are increasing.

Set-

$$\begin{aligned} F_R(x) &= \int_{-R}^{\infty} e^{ixt} d\alpha(t) \int_{-R}^{\infty} e^{ixt} d\beta(t) \\ &= 2^{-2xRi} \int_0^{\infty} e^{ixt} d\alpha(t-R) \int_0^{\infty} e^{ixt} d\beta(t-R) \end{aligned}$$

we have,

$$F_R(x) = e^{-2xRi} \int_0^{\infty} c^{ixt} d_{\varphi_R}(t) = \int_{-2R}^{\infty} d_{\varphi_R}(t+2R),$$

Where;

$$\begin{aligned} \varphi_R(t) &= \int_0^t \alpha(t-R-u) d\beta(u-R) \\ \varphi_R(t+2R) &= \int_{-R}^{t+R} \alpha(t-y) d\beta(y) \end{aligned} \quad (4.1,5)$$

Set-

$$\begin{aligned} \psi_R(t) &= \varphi_R(t+2R) && (-2R \leq t) \\ &= 0 && (t \leq -2R). \end{aligned}$$

Clearly, we are following the conventions of Chapter 4 in the definition of $\varphi_R(t)$ at a point t where the integral (4.1,5) is not defined. Then,

$$F_R(x) = \int_{-\infty}^{\infty} e^{ixt} d\psi_R(t) \quad (4.1,6)$$

From (4.1,5) for all t we consider,

$$\lim_{R \rightarrow \infty} \psi_R(t) = \int_{-\infty}^{\infty} \alpha(t-y) d\beta(y) = \gamma(t)$$

Assuming that R become non-finite under the integral sign of (4.1,6) our Axiom is formed. Thus, to analyze this point we use Axiom 4.1(a). From its definition it is obvious that $\psi_R(t)$ is a increasing function of t and that $\psi_R(-\infty) = 0$. Let us evaluate $\psi_R(\infty)$. Using, Axiom-

$$\gamma(-\infty) = 0$$

$$\gamma(\infty) = \alpha(\infty)\beta(\infty).$$

Now,

$$\psi_R(\infty) = \varphi_R(\infty) = \lim_{t \rightarrow \infty} \int_{-R}^{t+R} \alpha(t-y) d\beta(y).$$

Set-

$$\beta^*(y) = \beta(y) \quad y > -R$$

$$= \beta(-R) \quad y \leq -R$$

$$\alpha^*(y) = \alpha(y) \quad y > -R$$

$$= 0 \quad y \leq -R.$$

Then;

$$\int_{-R}^{t+R} \alpha(t-y) d\beta(y) = \int_{-\infty}^{\infty} \alpha^*(t-y) d\beta^*(y).$$

By Corollary-

$$\begin{aligned} \psi_R(\infty) &= \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} \alpha^*(t-y) d\beta^*(y) = \alpha^*(\infty) [\beta^*(\infty) - \beta^*(-\infty)] \\ &= \alpha(\infty) [\beta(\infty) - \beta(-R)] \end{aligned}$$

Hence,

$$\lim_{R \rightarrow \infty} \psi_R(\infty) = \alpha(\infty)\beta(\infty) = \gamma(\infty).$$

i.e., hypothesis of the Axiom is assured, and hence, proved.

4.2) COROLLARY:- If $\phi(t)$ and $\psi(t)$ belong to L in $(-\infty, \infty)$, then-

$$\int_{-\infty}^{\infty} e^{ixt} \phi(t) dt \int_{-\infty}^{\infty} \psi(t) dt = \int_{-\infty}^{\infty} e^{ixt} \omega(t) dt,$$

Where;

$$\omega(t) = \int_{-\infty}^{\infty} \phi(t-u)\psi(u) du.$$

If the theorem follows this, we take-

$$\alpha(t) = \int_{-\infty}^t \phi(u) du, \quad \beta(t) = \int_{-\infty}^t \psi(u) du.$$

Then,

$$\gamma(t) = \int_{-\infty}^{\infty} \psi(u) du \int_{-\infty}^{t-u} \phi(y) dy = \int_{-\infty}^{\infty} \psi(u) du \int_{-\infty}^t \phi(y-u) dy$$

Following Fubini's theorem-

$$\gamma(t) = \int_{-\infty}^t \omega(y) dy,$$

the result is formed. Using Fubini's theorem with no appeal to Axiom 4.1(a) it can be proved directly in quite a easy way.

V.CONCLUSION

The Laplace transform is used for solve the differential equation and integral equations. In physics and engineering the Laplace transform is used for analysis of linear time invariant system. In the kind of analysis, the Laplace transform is often assumed as a transform from the time domain, in which inputs and outputs are functions of complex angular frequency.

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