

Properties of c - β -Continuous Functions

Govindappa NAVALAGI

Department of Mathematics, KIT Tiptur, Karnataka, India

ABSTRACT: In 1970, Gentry and Hoyle have defined and studied the notion of c -continuity in topological spaces. Later, Long et al and Gauld have studied some more properties of c -continuity in the literature. In 1983, Abd El-Monsef et al have defined the concepts of β -open sets and β -continuous functions in topology, latter these sets were defined and studied as semipreopen sets by D.Andrijevic in 1986. In 1963 and 1982, respectively, defined the concepts of semiopen sets, semicontinuous functions & preopen sets and precontinuous functions in topological spaces. Recently, Navalagi defined and studied the concepts of c -precontinuous functions and c -semi-continuous functions as generalizations of c -continuous functions due to K.R.Gentry et al. In this paper, we define and study the concept of c - β -continuity. Also, we characterize their basic properties.

KEYWORDS: preopen sets, β -open sets, semiopen sets, compact subsets, β -continuity, c -continuity and M - β -continuity.

I. INTRODUCTION

In 1970, Gentry and Hoyle [10] have defined and studied the new class of functions called c -continuous functions. Later, in 1974 & 1975, Long et al [14 & 15] have studied further properties of c -continuous functions and defined a new class of functions called c^* -continuous functions in topological spaces. Again, in 1978 Gauld [9] has defined and studied some more properties of c -continuous functions via cocompact topologies. In 1982, Mashhour et al [16] have defined and studied the concepts of preopen sets and precontinuity in topology. In this paper, we define and study the concept of c -precontinuity. Also, we characterize their basic properties in topological spaces.

II. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply, X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f: X \rightarrow Y$ denotes a single valued function of a space (X, τ) into a space (Y, σ) . Let A be a subset of a space X . The closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space X is called regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$).

The following definitions and results are useful in the sequel:

Definition 2.1: A subset A of a space X is said to be:

- (i) α -open [24] if $A \subset Int(Cl(Int(A)))$
- (ii) semi-open [13] if $A \subset Cl(Int(A))$
- (iii) pre-open [16] if $A \subset Int(Cl(A))$

The family of all α -open (resp. semi-open, pre-open) sets in a space X is denoted by $\alpha O(X)$ (resp. $SO(X)$ $PO(X)$). The complement of an α -open (resp. pre-open) set is said to α -closed [18] (resp. pre-closed [8]).

Definition 2.2: The intersection of all preclosed sets containing A is called the preclosure of A and is denoted by $pClA$ [8].

The union of all pre-open sets contained in A is called pre-interior of A and is denoted by $pInt(A)$ [19].

Definition 2.3 [3&7]: A space X is called submaximal if each dense subset of X is open in X .

Lemma 2.4 [6 & 25]: A space X is submaximal iff every preopen (= dense) set is open.

Definition 2.3: A function $f: X \rightarrow Y$ is said to be:

- (i) precontinuous [16] , if the inverse image of each open subset of Y is preopen subset in X .
- (ii) p -continuous [20] , if the inverse image of each preopen subset of Y is an open subset in X .
- (iii) M -precontinuous [19] , if the inverse image of each preopen subset of Y is preopen subset in X ., equivalently, if the inverse image of each preclosed subset of Y is preclosed subset in X .

Definition 2.4 [21] : A function $f : X \rightarrow Y$ is said to be p -open (resp. p -closed) if the image of each preopen (resp. preclosed) subset of X is open (resp. closed) subset in Y .

Definition 2.5[19] : A function $f:X \rightarrow Y$ is said to be M -preopen(resp. M -preclosed) if the image of each preopen (resp. preclosed) subset of X is preopen (resp. preclosed) subset in Y .

Definition 2.6 [5] : A function $f : X \rightarrow Y$ is said to be presemiopen , if the image of each semiopen subset of X is semiopen in Y .

Definition 2.6[10] : A function $f : X \rightarrow Y$ is said to be c -continuous if for each $x \in X$ and each open set $V \subset Y$ containing $f(x)$ and having compact complement , there exists an open set U containing x such that $f(U) \subset V$.

Theorem 2.7[10,Th. 1] : Let $f : X \rightarrow Y$ be a function .Then the following statements are equivalent :

- (i) f is c -continuous.
- (ii)If V is an open subset of Y with compact complement , then $f^{-1}(V)$ is open subset of X .

These statements are implied by :

- (iii)If F is a compact subset of Y , then $f^{-1}(F)$ is closed subset of X and, moreover , if Y is Hausdorff, then all the above statements : (i)-(iii) are equivalent.

Theorem 2.8 : Let $f : X \rightarrow Y$ be a function . Then, f is c -continuous iff :

- (i)The inverse image of each open subset of Y having compact complement is open in X [10].
- (ii)The inverse image of each closed compact subset of Y is closed in X [14].

Definition 2.9[23] : A function $f : X \rightarrow Y$ is said to be c -precontinuous if for each $x \in X$ and each open set $V \subset Y$ containing $f(x)$ and having compact complement , there exists an preopen set U containing x such that $f(U) \subset V$.

Definition 2.10[4] : A function $f : X \rightarrow Y$ is said to be c -semicontinuous if for each $x \in X$ and each open set $V \subset Y$ containing $f(x)$ and having compact complement , there exists an semiopen set U containing x such that $f(U) \subset V$.

III. PROPERTIES OF C - β -CONTINUOUS FUNCTIONS

We, define the following.

Definition 3.1 : A function $f : X \rightarrow Y$ is said to be c - β -continuous if for each $x \in X$ and each open set $V \subset Y$ containing $f(x)$ and having compact complement , there exists an β -open set U containing x such that $f(U) \subset V$.

As , we know that every preopen or semiopen set is β -open set , so the following imply :

- (i) every c -continuous function is c - β -continuous.
- (ii) every c -precontinuous function is c - β -continuous .
- (iii) every c -semicontinuous function is c - β -continuous..

We , prove the following .

Theorem 3.2 : Let $f : X \rightarrow Y$ be a function .Then the following statements are equivalent :

- (i) f is c - β -continuous.

(ii) If V is an open subset of Y with compact complement, then $f^{-1}(V)$ is β -open subset of X .

These statements are implied by :

(iii) If F is a compact subset of Y , then $f^{-1}(F)$ is β -closed subset of X and, moreover, if Y is Hausdorff, then all the above statements : (i)-(iii) are equivalent.

Proof follows by Theorem 2.7 and 2.8 above.

Easy proof of the following is omitted.

Lemma 3.3 : A function $f : X \rightarrow Y$ is said to be c - β -continuous if the inverse image of each open subset of Y having compact complement is β -open in X .

Lemma 3.4 : A function $f : X \rightarrow Y$ is said to be c -precontinuous if the inverse image of each closed compact subset of Y is β -closed in X .

We, recall the following.

Lemma 3.5 [17] : If A is either open or α -open subset of X and V is a β -open subset of X , then $A \cap V$ is a β -open subset in the subspace $(A, \tau/A)$.

Next, we prove the following.

Theorem 3.6 : If $f : X \rightarrow Y$ is c - β -continuous function and A be an either open or α -open subset of X , then $f/A : A \rightarrow Y$ is also c - β -precontinuous.

Easy proof of the Theorem follows by Lemma – 3.5 above.

We, define the following.

Definition 3.7 : A function $f : X \rightarrow Y$ is said to be M - β -continuous, if the inverse image of each β -open subset of Y is β -open subset in X , equivalently, if the inverse image of each β -closed subset of Y is β -closed subset in X .

Theorem 3.8: If $f : X \rightarrow Y$ is M - β -continuous and $g : Y \rightarrow Z$ is c - β -continuous, then $g \circ f$ is c - β -continuous.

Proof. Let U be an open subset of Z having compact complement. Then, $g^{-1}(U)$ is β -open set in Y , since g is c - β -continuous. Again, as f is M - β -continuous and $g^{-1}(U)$ is β -open subset of Y , $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is β -open subset in X . This shows that $g \circ f$ is c - β -continuous function.

We, define the following.

Definition 3.9 : A function $f : X \rightarrow Y$ is said to be β^* -continuous, if the inverse image of each β -open subset of Y is open subset in X .

Theorem 3.10: If $f : X \rightarrow Y$ is β^* -continuous and $g : Y \rightarrow Z$ is c - β -continuous, then $g \circ f$ is c -continuous function.

Proof follows from Theorem-3.6.

We, define the following.

Definition 3.11 : A function $f : X \rightarrow Y$ is called M - β -open (resp. M - β -closed) if the image of each β -open (resp. β -closed) subset of X is β -open (resp. β -closed) subset in Y .

Theorem 3.11 : Let $f : X \rightarrow Y$ be either M - β -open or M - β -closed surjective function and let $g : Y \rightarrow Z$ be any function such that $g \circ f$ is c - β -continuous. Then, g is c - β -continuous.

Proof : Suppose f is M - β -open (resp. M - β -closed) and V be an open subset with compact complement (resp. V be a closed compact subset) in Z . Since $g \circ f$ is c - β -continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is β -open (resp. β -closed) subset in X .

X . Since f is M - β -open (resp. M - β -closed) and surjective, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is β -open (resp. β -closed) set in Y and consequently, g is c - β -continuous function.

We, define the following.

Definition 3.12: A function $f : X \rightarrow Y$ is said to be β^* -open (resp. β^* -closed), if the image of each β -open (resp. β -closed) subset of X is open (resp. closed) subset in Y .

Theorem 3.13 : Let $f : X \rightarrow Y$ be either β^* -open or β^* -closed surjection and let $g : Y \rightarrow Z$ be any function such that $g \circ f$ is c - β -continuous. Then, g is c -continuous.

Proof follows by Theorem -3.8 above.

In view of the fact that an arbitrary union of α -open sets is α -open, we have the following [7 , 10 & 12].

Theorem 3.14 : If X and Y are two topological spaces and $X = A \cup B$, where A and B are open or α -open subsets of X and $f : X \rightarrow Y$ is a function such that $f|_A$ and $f|_B$ are c - β -continuous, then f is c - β -continuous.

Proof : Assume that A and B are open or α -open subsets in X . Let U be an open subset of Y with compact complement. Then, we have $f^{-1}(U) = (f|_A)^{-1}(U) \cup (f|_B)^{-1}(U)$, each of which is β -open by Lemma-3.5 & Theorem- 3.6. Thus, $f^{-1}(U)$ is β -open in X and hence f is c - β -continuous.

Recall that a space X is called submaximal [2] if every dense subset of X is open. Also, it is proved in [1] that a space X is submaximal iff every preopen set in it open.

In view of the above quoted submaximal space and result, we give the following.

Theorem 15: Let X be a submaximal space and let Y be a locally compact regular space. If $f : X \rightarrow Y$ is c - β -continuous then f is continuous.

Recall that a space X is called β - T_1 [6] if, for $x, y \in X$ such that $x \neq y$, there exist β -open sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Also, it is proved that in [6 & 16] a β - T_1 space every singleton set is β -closed.

In view of the above result, we give the following.

Theorem 16 : Let $f : X \rightarrow Y$ be c - β -continuous and injective. If Y is T_1 , then X is β - T_1 .

We, recall the following.

Definition 3.19[11]: Let $f : X \rightarrow Y$ be a function. Then, $G(f) = \{(x, f(x)) \mid x \in X\}$ is called the graph of f and the function $g(f) : X \rightarrow X \times Y$ defined as $g(f)(x) = (x, f(x))$ for each $x \in X$ is called the graph function of f .

Definition 3.20 [2] : A function $f : X \rightarrow Y$ is said to have preclosed graph, if for each pair $y \neq f(x)$, there exists a preopen set V of Y containing y and a preopen set U containing $x \in X$ such that $f(U) \cap V = \emptyset$.

Next, we prove the following.

Theorem 3.21: Let Y be a locally compact Hausdorff space. Then, c -precontinuous function $f : X \rightarrow Y$ has a preclosed graph.

Proof : Let $(x, y) \notin G(f)$. Then, $y \neq f(x)$. As, Y is locally compact Hausdorff space, so there exists an open set V containing y such that $f(x) \notin \text{Cl}(V)$, where $\text{Cl}(V)$ is compact in Y . Now, $f(x) \in Y \setminus \text{Cl}(V)$, i.e., $x \in f^{-1}(Y \setminus \text{Cl}(V))$. Since f is c -precontinuous, there exists a preopen set U containing x such that $x \in U \subset f^{-1}(Y \setminus \text{Cl}(V))$. Thus, $f(U) \cap \text{Cl}(V) = \emptyset$, i.e., $f(U) \cap V = \emptyset$. This shows that f has a preclosed graph.

Theorem 3.22 : Let $f : X \rightarrow Y$ be c -precontinuous. Then, the graph function $g(f) : X \rightarrow X \times Y$ is c -precontinuous.

Proof : Let $U \times V$ be any open subset in $X \times Y$ having compact complement W of $X \times Y$. Then, we have to show that $(g(f))^{-1}(U \times V)$ is preopen set in X . Let $W = X \times Y \setminus (U \times V) = (X \setminus U) \times Y \cup X \times (Y \setminus V)$, in which $X \times (Y \setminus V)$ being the closed subset of W must also be compact. Since $P_Y : X \times Y \rightarrow Y$ being the projection, which is continuous, so $P_Y(X \times (Y \setminus V)) = Y \setminus V$ is compact in Y . Thus, $f^{-1}(V)$ is preopen set in X . Since f is c -precontinuous, $(g(f))^{-1}(U \times V) = U \cap f^{-1}(V)$, which is preopen as the intersection of an open set and a preopen set is again preopen. Therefore, $g(f)$ is c -precontinuous.

Theorem 3.23: Let X be compact Hausdorff space. If $g(f) : X \rightarrow X \times Y$ is c -precontinuous, then the function $f : X \rightarrow Y$ is c -precontinuous.

Proof : Let V any open set containing $f(x)$ having compact complement. Then, we have to prove that $f^{-1}(V)$ is preopen in X : Consider $X \times V$ which is open in $X \times Y$ where $X \times Y \setminus (X \times V) = X \times (Y \setminus V)$ is compact, and $g(f)$ is c -precontinuous and hence $(g(f))^{-1}(X \times V) = f^{-1}(V)$ which is preopen in X . This shows that f is c -precontinuous.

IV. CONCLUSION

In 1970, K.R.Gentry et. al. have defined and studied the notion of c -continuous functions in topology. Later, in 2001 G.I.Chae et.al. have studied the notion of c -semicontinuous functions in topology. In 2014 and 2018, Navalagi studied the notions of c -precontinuous functions and c - α -continuous functions in topology. Our aim is study some more properties of c - β -continuous functions in the future papers.

REFERENCES

- [1] T.Aho and T.Nieminen, Spaces in which preopen subsets are semiopen, *Ricerche Mat.*, 43 (1994), 45-59.
- [2] N.Bandyopadhyay and P.Bhattacharyya, Functions with preclosed graphs, *Bull. Malaysian Math.Sci.Soc.*, (2) 28(1) (2005), 87-93.
- [3] N.Bourbaki, *General Topology, Part-I*, Addison-Wesley, Reading Mass (1966).
- [4] G.I.Chae, R.K.Pooniwala and V.P.Singh, A note on c -semicontinuous functions, *KORUS*, 2001, 110-112.
- [5] S.G.Crossley and S.K.Hildebrand, Semitopological properties, *Fund.Math.*, 74, (1972), 233-254.
- [6] J.Dontchev, Survey on preopen sets, arXiv: math/9810177, Volume.1 [math.GN], 30th Oct-1998, 1-18.
- [7] J.Dontchev, M.Ganster and T.Noiri, On p -closed spaces, *Inter. J. Math. & Math. Sci.*, Vol.24, no. 3 (2000), 203 – 212.
- [8] S.N.El-Deeb, I.A. Hasanein, A.S.Mashhour and T.Noiri, On p -regular spaces, *Bull. Math.Soc.Sci., Math., R.S.Roumanie (N.S)*, 27(75), (1983), 311-315.
- [9] D.B.Gauld, c -continuous functions and cocompact topologies, *Kyungpook Math. J.*, Vol.18, No.2, Dec-1978, 151-157.
- [10] K.R.Gentry and H.B.Hoyle, III, c -continuous functions, *Yokohama Math. J.*, 18 (1970), 71-76.
- [11] T.Husain, *Topology and maps*, Plenum Press, New York, 1977.
- [12] A.Kar and P.Bhattacharyya, Some weak separation axioms, *Bull.Calcutta Math. Soc.*, 82(1990), 415-422.
- [13] N.Levine, Semiopen sets and semicontinuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.
- [14] P.E.Long and M.D.Hendrix, Properties of c -continuous functions, *Yokohama Math. J.*, 22(1974), 117-123.
- [15] P.E.Long and L.L.Herrington, Properties of c -continuous functions and c^* -continuous functions, *Kyungpook Math. J.*, Vol.15, No.2, Dec-1975, 213-221.
- [16] A.S.Mashhour, M.E.Abd El-Monsef and S.N.El-Monsef, On pre-continuous and weak precontinuous mappings, *Proc.Math. Phys. Egypt*, 53 (1982), 47-53.
- [17] A.S.Mashhour et al, A note on semicontinuity and precontinuity, *Indian J. pure appl. Math.*, 13(10), Oct- 1982, 1119-1123.
- [18] A.S.Mashhour, I.A.Hasanein and S.N.El-Deeb, α -continuous and α -open mappings, *Acta Math.Hungarica*, 41(3-4)(1983), 213-218.
- [19] A.S.Mashhour, M.E.Abd El-Monsef and I.A.Hasanein, On pre topological spaces, *Bull. Math. Soc. Sci. Math., R.S.Roumanie*, 28(76)(1984), 39-45.
- [20] G.B.Navalagi, Completely preirresolute functions and completely gp -irresolute functions, *Inter.J. of mathematics and Computing Applications*, Vol.3, Nos.1-2, Jan-Dec. 2011, 55-65.
- [21] Govindappa Navalagi, Generalization of preirresolute functions, *Inter. J.of General Topology*, Vol.3, Nos.1-2, Jan-Dec. 2010, 65-71.



- [22] Govindappa Navalagi, Further properties of pre-T_0 , pre-T_1 and pre-T_2 spaces, Inter. J. of Mathematics and Computing Applications, Vol.3, Nos.1-2, Jan-Dec. 2011, 67-75.
- [23] Govindappa Navalagi, On c -precontinuous functions, American J. of Mathematics and Sciences, Vol.3, No.1, (January 2014), 41-48.
- [24] O.Njastad, On some classes of nearly open sets, Pacific J.Math., 15 (1965), 961-970.
- [25] I.L.Reilly and M.K.Vamanamurthy, On some questions concerning preopen sets, Kyungpook Math.J., 30(1990), 87-93.