

Network Analysis Having Arbitrary Resistances and Voltages with the Help of Graph Theory

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ABSTRACT: Now-a-days Graph theory, a branch of Mathematics has impact on almost every other branch. In engineering mathematics also, graph theory plays a vital rule. Study of the properties of electrical circuits with the help of graph theory is a growing trend in mathematical and electrical fields. Electrical circuits consists of nodes and branches which obeys Kirchhoff's current laws, Kirchhoff's voltage laws etc. The scientists now-a-days engage themselves to construct simple cost effective circuits. There are various well known theorems such as Norton's theorem, Thevenin's theorem, Superposition theorem, Millman Theorem etc to construct simple circuits from the complicated circuits. In the graph of the circuits, all the elements of the circuits are replaced by lines with circles or dots at both ends. The lines represent branches and the dots represent nodes of the circuits. From these graphs, we can find matrices related to them by different rules. In this paper, we try to study Millman's theorem with the help of Graph theorem.

KEYWORDS : Graph theory, Electrical circuits, Branch, Node, Matrix Representation, Branch current, Loop current, Adjacency matrix, Incident matrix, Impedance matrix, Branch admittance matrix

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I. INTRODUCTION

Graph theory is the branch for the study of proof techniques in discrete mathematics and its results have application in many areas of the computing, social and natural sciences. Spectral graph theory is the study of the properties of a graph which describes the relation between the polynomial eigenvalues and eigenvectors of matrices associated with the graph.

A. GRAPH: A graph is a set of ordered pair $G = (V, E)$ of sets where $E = \{\{x, y\} : x, y \in V\}$. The elements of V are called vertices (or nodes) of the graph G and the elements of E are called edges. So in a graph a vertex set is a set of points $\{x_1, x_2, x_3 \dots x_n\}$ and edge is a line which connects two points x_i and x_j . Such graphs are called undirected graphs. A directed graph is similar to an undirected graph except the edge set $E \in V \times V$.

B. ELECTRICAL CIRCUITS : An electrical circuit consists of internally connected elements viz resistors, capacitors, inductors, diodes, transistors etc. The behaviour of an electrical circuits generally depends upon two factors.

- a) The characteristic of each of internally connected elements
- b) The rule by which they are connected together.

The second factor gives a relation between electrical circuits with graph theory. A two terminal electrical element can be represented by an edge e_k . Associate with each edge there are two

variables $V_k(t)$ and $i_k(t)$. The variable $V_k(t)$ is called the edge voltage and may be regarded as cross variable because it exist across the two end points. The other variable is called edge current and may be regarded as through variable because it flows through the edge. These variables must also obey the two laws of Kirchoff's.

C. KIRCHHOFF'S CURRENT LAW (KCL): For any lumped electrical network, at any time the net sum (taking into account the orientations) of all the currents leaving any node or vertex is zero. That is at r^{th} vertex of the corresponding digraph , we must have

$$\sum_{k=1}^e a_{rk} i_k(t) = 0$$

Where a_{rk} is the rk^{th} entry of the incidence matrix A of G and $i_k(t)$ is the amount of current flowing through the k^{th} edge of G.

D. KIRCHHOFF'S VOLTAGE LAW (KVL) : For any lumped electrical network, at any time the net sum (taking into account the orientations) of the voltages around a loop (i.e. circuit) is zero. In terms of the corresponding digraph , for the r^{th} circuit we must have

$$\sum_{k=1}^e b_{rk} v_k(t) = 0$$

Where b_{rk} is the rk^{th} entry of the circuit matrix B of G and $V_k(t)$ is the amount of voltage across the k^{th} edge.

E. FROM CIRCUIT TO GRAPH :

A graph can be obtained from a circuit. We identify the graph $G=(V, E)$ where V is the set of vertices and E is the set of edges. The edge between i^{th} and j^{th} vertices can be denoted by $\{i,j\}$ ignoring the direction. Similarly the notation (i,j) can be used for oriented edges, where i is the start vertex and j is the end vertex. For example consider the circuit and its graph in the figure. There are five vertices and seven edges in the graph obtained from the given circuit. An edge is sometimes called branch and a vertex is called node in case of electrical circuits.

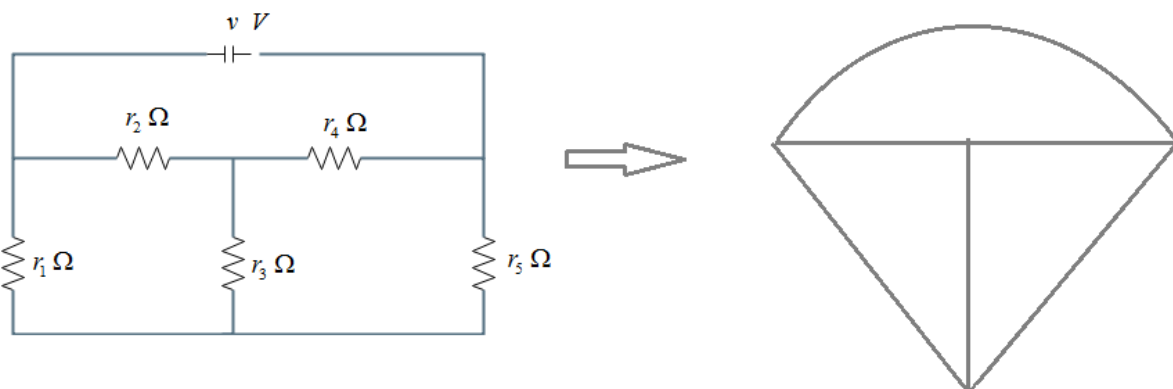


Fig.1 : A circuit and its corresponding graph

F. MATRICES ASSOCIATED TO A GRAPH :

(a) **Adjacency matrix:**

Let G be a graph without multiple edges. The *adjacency matrix* is the 0-1 matrix A indexed by the vertex set V of G , where $A_{xy}= 1$ when there is an edge from x to y in G and $A_{xy}= 0$ otherwise. In multi graphs A_{xy} equals the number of edges from x to y.

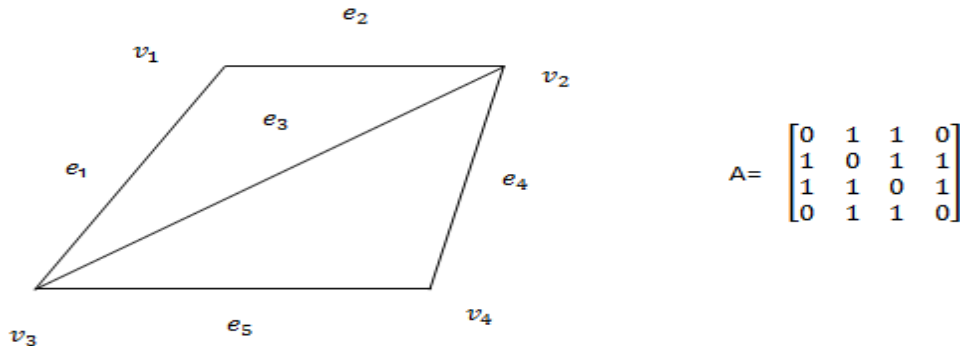


Fig.2 : A matrix and its adjacency matrix

(b) Incidence matrix:

Let G be an undirected graph without loops. The (vertex-edge) *incidence matrix* of G is the 0-1 matrix M, where rows represent the vertices and columns represent the edges, where $M_{xe} = 1$ when vertex x is an endpoint of edge e. Let G be a directed graph without loops. The *directed incidence matrix* of G is the 0-1 matrix N, with rows represent vertices and columns represent edges, where $N_{xe} = -1, 1, 0$ when e starts at x, e ends at x, or otherwise respectively.

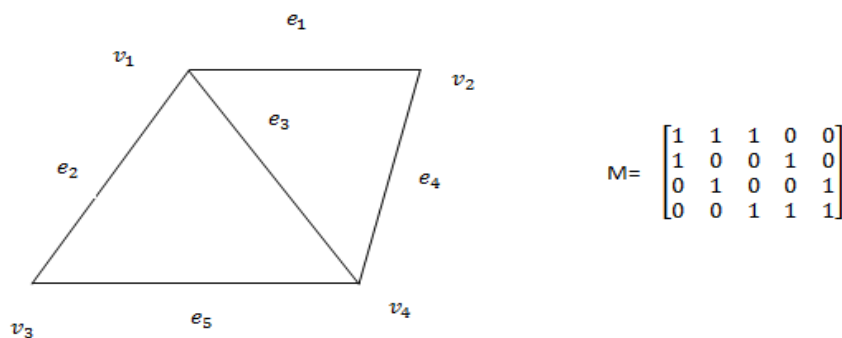


Fig. 3: A graph and its Incidence matrix

Fig. 4: A graph and its directed incidence matrix

(c) Fundamental Tie set matrix (Fundamental loop matrix) :

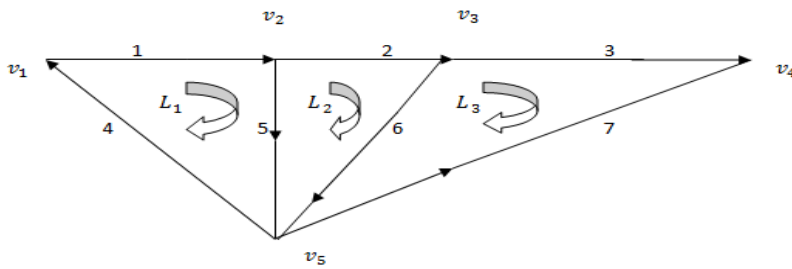
This matrix is associated to a fundamental loop i.e. a loop formed by only one link (branch that does not belong to a particular tree) associated with other twigs (branch of trees). Here we assume that the direction of loop currents and direction of the link is same.

So in the matrix

$$b_{ij} = 1, \text{ if the branch } b_j \text{ in the fundamental loop } i \text{ and their reference direction oriented same}$$

$$b_{ij} = -1 \text{ if the branch } b_j \text{ in the fundamental loop } i \text{ and their reference direction oriented opposite}$$

$$b_{ij} = 0 \text{ if the branch } b_j \text{ is not in the fundamental loop } i$$



Here, we consider three loops. L_1 consists of branches 1, 5, 4, L_2 consist of branches 2, 5, 6 L_3 Consists of branches 3, 6, 7. Direction of all the loops is taken anticlockwise. So in the matrix where loops are along the rows and branches are along the columns is given by

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{bmatrix}$$

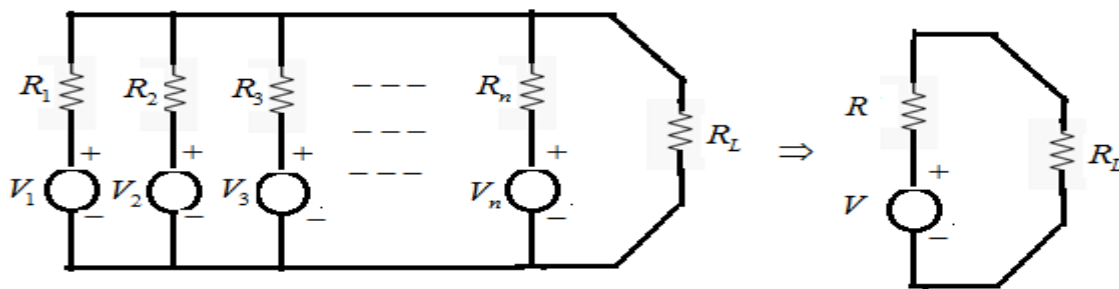
(d) **Branch Impedance matrix $[Z_b]$:**

It is a square matrix of order m where m is the no of branches having branch impedance as the diagonal elements and mutual impedance as off diagonal elements. If there is no transformer or mutual sharing then off diagonal entries are zero.

II. MILLMAN'S THEOREM

With the help of this theorem any numbers of parallel voltage sources can be reduce to one equivalent source .

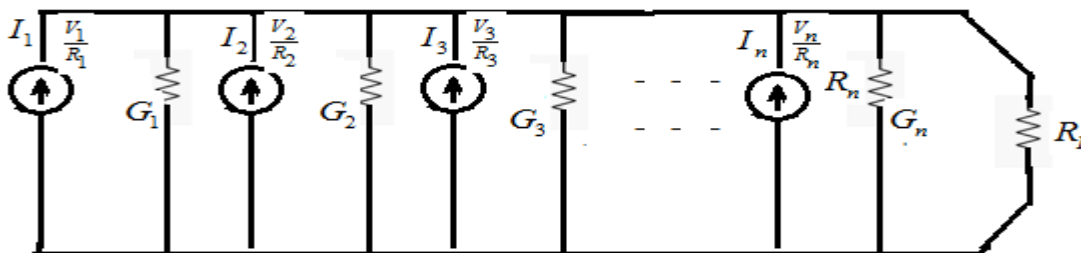
Let us consider a number of parallel voltage sources $V_1, V_2, V_3, \dots, V_n$ having internal resistances $R_1, R_2, R_3, \dots, R_n$ respectively .The arrangement can be replaced by a single equivalent voltage source V in series with an equivalent series resistance R as shown in the figure .



By Millman's theorem the equivalent voltage and resistance of the reduced circuit is given by the relations

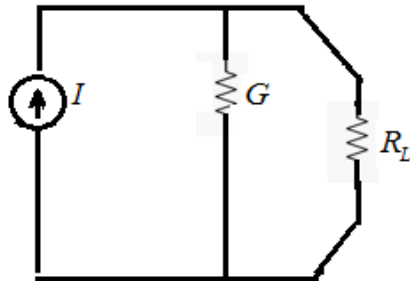
$$V = \frac{\pm V_1 G_1 \pm V_2 G_2 \pm V_3 G_3 \pm \dots \pm V_n G_n}{G_1 + G_2 + G_3 + \dots + G_n} \quad \text{where} \quad R = \frac{1}{G} = \frac{1}{G_1 + G_2 + G_3 + \dots + G_n}$$

Let us convert the voltage sources into current sources as follows

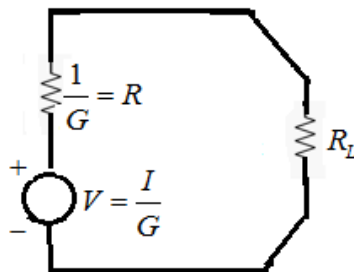


If I be the resultant current of the parallel current sources where G is the equivalent conductance .
 $I = I_1 + I_2 + I_3 + \dots + I_n$ and $G = G_1 + G_2 + G_3 + \dots + G_n$.

Then the equivalent circuit will be



Now converting the current sources to equivalent voltage sources as



Thus we have $V = \frac{I}{G} = \frac{\pm I_1 \pm I_2 \pm I_3 \pm \dots \pm I_n}{G_1 + G_2 + G_3 + \dots + G_n}$

Also $R = \frac{1}{G} = \frac{1}{G_1 + G_2 + G_3 + \dots + G_n}$

and $V = \frac{\pm \frac{V_1}{R_1} \pm \frac{V_2}{R_2} \pm \frac{V_3}{R_3} \pm \dots \pm \frac{V_n}{R_n}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots + \frac{1}{R_n}}$

Where R is the equivalent resistance connected with the equivalent voltage source in series .

III. OUR PROBLEM

With the help of Millman's theorem find the current through $r_4 \Omega$ of the following circuit .

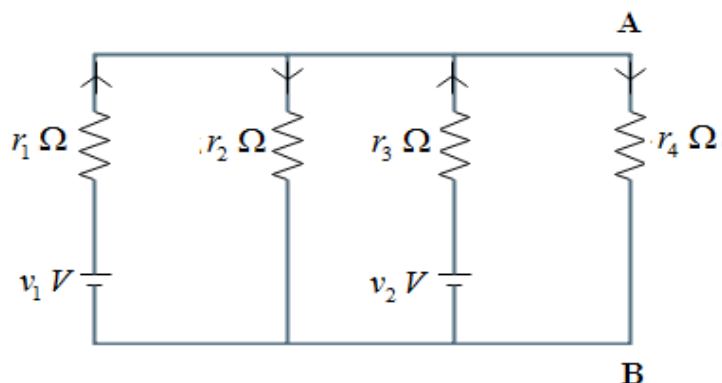


Fig. 5: Circuit having arbitrary resistances and voltages

Voltage across the terminal $A-B = V_{AB} = \frac{\frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{v_1 + 0 + v_2}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = \frac{r_2(r_3v_1 + r_1v_2)}{r_1r_2 + r_2r_3 + r_3r_1}$

And equivalent resistance $R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = \frac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_3r_1}$ ohms

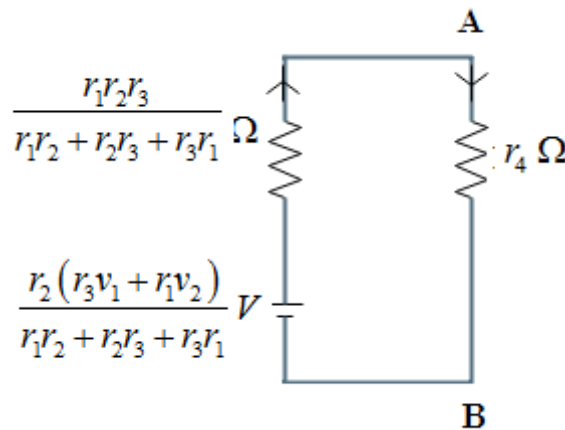


Fig. 6: Simplified circuit

Therefore by Millman's theorem the current through $r_4 \Omega$ is obtained as

$$I_{r_4\Omega} = \frac{V_{Th}}{R_{eq} + r_4} = \frac{\frac{r_2(r_3v_1 + r_1v_2)}{r_1r_2 + r_2r_3 + r_3r_1}}{\frac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_3r_1} + r_4} = \frac{r_2(r_3v_1 + r_1v_2)}{r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2} A$$

Now let us study this circuit with the help of graph theory .

The graph of the given circuit is as given below .This graph has two vertices 1 and 2 and four edges .

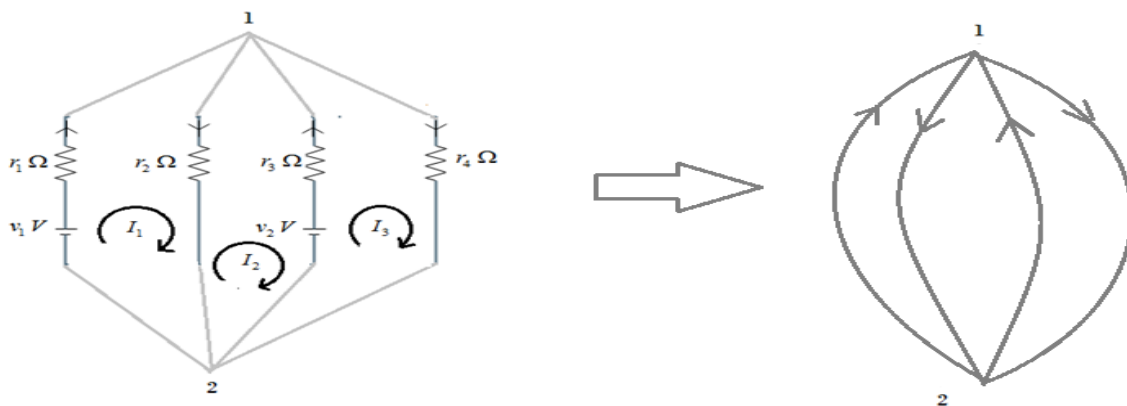


Fig.7: Graph of the circuit.

$$[B] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad [V_s] = \begin{bmatrix} v_1 \\ 0 \\ v_2 \\ 0 \end{bmatrix} \quad [Z_b] = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}$$

$$[B][V_s] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \\ v_2 \end{bmatrix}$$

$$[B][Z_b] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & 0 & 0 \\ 0 & -r_2 & -r_3 & 0 \\ 0 & 0 & r_3 & r_4 \end{bmatrix}$$

$$[B][Z_b][B'] = \begin{bmatrix} r_1 & r_2 & 0 & 0 \\ 0 & -r_2 & -r_3 & 0 \\ 0 & 0 & r_3 & r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_1 + r_2 & -r_2 & 0 \\ -r_2 & r_2 + r_3 & -r_3 \\ 0 & -r_3 & r_3 + r_4 \end{bmatrix}$$

By Kirchhoff 's Voltage Law we have

$$[B][Z_b][B'] [I] = -[B][V_s]$$

$$\Rightarrow \begin{bmatrix} r_1 + r_2 & -r_2 & 0 \\ -r_2 & r_2 + r_3 & -r_3 \\ 0 & -r_3 & r_3 + r_4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = - \begin{bmatrix} v_1 \\ -v_2 \\ v_2 \end{bmatrix}$$

$$\therefore (r_1 + r_2)I_1 - r_2I_2 = -v_1$$

$$-r_2I_1 + (r_2 + r_3)I_2 - r_3I_3 = v_2$$

$$-r_3I_2 + (r_3 + r_4)I_3 = -v_2$$

$$\therefore I_1 = \frac{-r_2r_3v - r_2r_4v - r_3r_4v_1 + r_2r_4v_2}{r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2} A$$

$$I_2 = \frac{-r_2r_3v_1 - r_2r_4v_1 + r_1r_4v_2 + r_2r_4v_2}{r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2} A$$

$$I_3 = - \frac{r_2r_3v_1 + r_1r_2v_2}{r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2} A$$

Now we have to find the branch current of the circuit

$$[i_b] = [B'] [I_L]$$

$$\Rightarrow \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-r_2 r_3 v_1 - r_2 r_4 v_1 - r_3 r_4 v_1 + r_2 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \\ \frac{-r_2 r_3 v_1 - r_2 r_4 v_1 + r_1 r_4 v_2 + r_2 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \\ \frac{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \\ \frac{r_2 r_3 v_1 + r_1 r_2 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} \frac{-r_2 r_3 v_1 - r_2 r_4 v_1 - r_3 r_4 v_1 + r_2 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \\ \frac{-r_3 r_4 v_1 - r_1 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \\ \frac{r_2 r_4 v_1 - r_1 r_2 v_2 - r_1 r_4 v_2 - r_2 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \\ \frac{-r_2 r_3 v_1 - r_1 r_2 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2} \end{bmatrix}$$

$$\therefore i_1 = \frac{-r_2 r_3 v_1 - r_2 r_4 v_1 - r_3 r_4 v_1 + r_2 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}$$

$$i_2 = \frac{-r_3 r_4 v_1 - r_1 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}$$

$$i_3 = \frac{r_2 r_4 v_1 - r_1 r_2 v_2 - r_1 r_4 v_2 - r_2 r_4 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}$$

$$i_4 = \frac{-r_2 r_3 v_1 - r_1 r_2 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}$$

Therefore current across the resistance $r_4 \Omega$ is $i_4 = \frac{-r_2 r_3 v_1 - r_1 r_2 v_2}{r_1 r_2 r_3 + r_2 r_3 r_4 + r_3 r_4 r_1 + r_4 r_1 r_2}$, which is same as the current obtain by Millman's theorem.

We see that the amount of current in the simplified circuit is same as the current obtained by Millman's equivalent circuit .(Negative sign shows that orientation of the current is in opposite direction)

IV. CONCLUSION

From the above study , it is observed that the result obtained by Millman's theorem is same as the result obtained by graph theoretic approach using network equilibrium equations. Following this

concept one can develop the equivalent circuits from the complicated circuits with the help of graph theory

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