

A Significant Condition for Solving Moment Problem

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ABSTRACT : This present paper deals with the moment problem. The Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval. R.P. Boas [1939] has observed that in this case there is hardly any problem, as every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution [2]. condition for solving moment problem and determining and non-determining solution.

KEYWORDS : Moment Problem, Stieltjes problem, Hamburger moment problem, Inequalities, Moment sequence, Laplace transform.

I. INTRODUCTION

1.1 Moment Problem

As per engineering mechanicals tools and technique, When a force of certain magnitude is applied on a particle then moment is produce, mathematically, this is measured as the product of force applied and the effective distance of the shape and size of the particle or substance taken for the pupose. Simple motto behind explaining this definition is that moment is govern by two variables and so there should be limitation of exact two variables in case of moment based problems. This fact is an eyes opener for those who consider only one variable and and talk about existence of a Moment Problem only.

II RELATED WORK

When μ is a positive measure with moments $(s_n)_{n \geq 0}$ we say that μ is a solution to the moment problem. If the solution to the moment problem is unique, the moment problem is called determinate. otherwise the moment problem is said to be indeterminate [1]. The moment problem on $[0,1)$ is referred to as the Hausdorff's moment problem and the moment problem on \mathbb{R} is called the Hamburger moment problem and thus $[0,\infty)$ is called the Stieltjes moment problem [12].

Statement of the moment problem is given as below-

$$\{\mu_n\}_0^\infty : \mu_0, \mu_1, \mu_2, \dots; \tag{1.1,1}$$

There are other forms of existence to predict function $\alpha(t)$ of bounded variation in interval $(0,1)$ for example -

$$\mu_n = \int_0^1 t^n d\alpha(t) \quad (n = 0, 1, 2, \dots) \tag{1.1,2}$$

Hence, this is corollary known as moment sequence. But, it has been noticed that every sequence (1.1,1) has them form (1.1,2) since (1.1,2) implies that-

$$|\mu_n| \leq V[\alpha(t)]_0^1$$

the quantity on the right being the variation of $\alpha(t)$ on the interval $(0, 1)$. Thus, in this way its sequence is confined. It was F. Hausdorff [1921a] to determine essential situation of sequence to be moment sequence. But, given that representation (1.1,2) if $\alpha(t)$ is a normalized function of corollary differences.

$$\alpha(0) = 0, \quad \alpha(t) = \frac{\alpha(t+) + \alpha(t-)}{2} \quad (0 < t < 1).$$



As normalization of the function $\alpha(t)$ does not varies the value of the integral (1.1,2) so, we can take over the loss with any disturbance to general prospect $\alpha(t)$ which is normalized. Infact, the entire discussion throw light upon it without any obstacle.

Equations (1.1,2) may be admired as an inter-changeability of the function $\alpha(t)$ into the sequence $\{\mu_n\}$. This inter-changeability is closely attached to the Laplace transform, is in fact the discrete analogue of the latter. For, if we substitute the integer n by the variable s in (1.1,2) and then make the change of variable $t = e^{-u}$, we obtain-

$$\mu_s = \int_0^\infty e^{-su} d[-\alpha(e^{-u})].$$

Hence, it is perfectly observed that enough importance relays on `Laplace transform` by a solution of the moment problem.

III. METHODOLOGY

2.1 The Hamburger Problem

The Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval [13]. R.P. Boas [1939] has observed that in this case there is hardly any problem, as every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution. We provide the proof of Boas. which will of course be enough to initialize the Stieltjes case. Axiom 2.1(a) The equations

$$\mu_n = \int_0^\infty t^n d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

For ever have a solution $\alpha(t)$ of closely related variation for which

$$\int_0^\infty |d\alpha(t)| < \infty.$$

We formed the other two sequences $\{\lambda_n\}_0^\infty, \{v_n\}_0^\infty$, such as;

$$\mu_n = \lambda_n - v_n \quad (2.1,1)$$

$$\lambda_n = \int_0^\infty t^n d\beta(t) \quad (2.1,2)$$

$$v_n = \int_0^\infty t^n d\gamma(t) \quad (n = 0, 1, 2, \dots), \quad (2.1,3)$$

Now, here $\beta(t)$ and $\gamma(t)$ are closely related increasing functions. First choose $\lambda_0, \lambda_1, v_0, v_1$ as any non-negative numeric satisfying (2.1,1). Proceeding by induction, Let, λ_k, v_k for $k = 0, 1, 2, \dots, 2n - 1$ so that (2.1,1) holds and so that the determinants

$$[\lambda_0, \lambda_1, \dots, \lambda_{2k}] = \begin{vmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_k \\ \lambda_1 & \lambda_2 & \dots & \lambda_{k+1} \\ \dots & \dots & \dots & \dots \\ \lambda_k & \lambda_{k+1} & \dots & \lambda_{2k} \end{vmatrix} \quad (2.1,4)$$

$$[\lambda_1, \lambda_2, \dots, \lambda_{2k+1}] = \begin{vmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{k+1} \\ \lambda_2 & \lambda_3 & \dots & \lambda_{k+2} \\ \dots & \dots & \dots & \dots \\ \lambda_{k+1} & \lambda_{k+2} & \dots & \lambda_{2k+1} \end{vmatrix},$$

$$[v_0, v_1, \dots, v_{2k}], [v_1, v_2, \dots, v_{2k+1}] \quad (2.1,5)$$

are non-negative for $k = 0, 1, \dots, n - 1$. Thus, we explain & define $\lambda_{2n}, v_{2n}, \lambda_{2n+1}, v_{2n+1}$. We have undetermined λ_{2n}

$$[\lambda_0, \lambda_1, \dots, \lambda_{2n}] = \lambda_{2n} [\lambda_0, \lambda_1, \dots, \lambda_{2n-2}] + P, \quad (2.1,6)$$



Here, P is a polynomial in $\lambda_0, \lambda_1, \dots, \lambda_{2n-1}$; and similarly for $[v_0, v_1, \dots, v_{2n}]$. Since zero is smaller than $[\lambda_0, \lambda_1, \dots, \lambda_{2n-2}]$ and $[v_0, v_1, \dots, v_{2n-2}]$ so, we can choose λ_{2n} and v_{2n} non-negative and so large that $\lambda_{2n} - v_{2n} = \mu_{2n}$ as-

$$[\lambda_0, \lambda_1, \dots, \lambda_{2n}] > 0, \quad [v_0, v_1, \dots, v_{2n}] > 0$$

It has been observed that (2.1,6) holds with all subscripts increased by unity, P now being a polynomial in $\lambda_1, \lambda_2, \dots, \lambda_{2n}$, a similar equation holding for the v_k . With λ_{2n} and v_{2n} now determined we proceed exactly as above to determine λ_{2n+1} and v_{2n+1} . Thus, the induction is completed. Following Axiom 2.1(b) if the determinants (2.1,4) and (2.1,5) are non-negative for $k = 0, 1, 2, \dots$ equations (2.1,2) and (2.1,3) have closely related solutions $\beta(t)$ and $\gamma(t)$ respectively, so that when $\alpha(t)$ is explained & defined as $\beta(t) - \gamma(t)$ our proof is proved.

Stieltjes is mentioned to prove there exists a function, not a constant, all the moments of which are zero. It too follows from Axiom 2.1(a). For, by this result, there is an existence of a non-constant function $\alpha(t)$ such as-

$$\int_0^\infty t^n d\alpha(t) = 1 \quad (n = 1)$$

$$(n = 0, 2, 3, 4, \dots)$$

$$\int_0^\infty |d\alpha(t)| < \infty.$$

Putting;

$$\beta(t) = \alpha(t^{1/2}),$$

we form-

$$\int_0^\infty t^n d\beta(t) = \int_0^\infty t^{2n} d\alpha(t) = 0 \quad (n = 0, 2, 3, 4, \dots).$$

The function $\beta(t)$ is the required example.

IV. RESULTS AND DISCUSSION

3.1 The condition for solving moment problem

Boas showed that any sequence which increases sufficiently rapidly leads to a soluble Stieltjes problem [with increasing $\alpha(t)$] by a slight modification of the method employed. More precisely, the outcome is

Axiom 3.1(a) If

$$\mu_0 \geq 1, \quad \mu_n \geq (n\mu_{n-1})^n \quad (n = 1, 2, \dots) \quad (3.1,7)$$

So, the equations

$$\mu_n = \int_0^\infty t^n d\alpha(t) \quad (n = 0, 1, 2, \dots)$$

have an increasing solution $\alpha(t)$

For example- Given sequence satisfying (2.1, 1) is $\mu_0 = 1, \mu_n = n^{n^n}$ for $n = 1, 2, \dots$ as considered

$$[\mu_0, \mu_1, \dots, \mu_{2n}] = \mu_{2n} [\mu_0, \mu_2, \dots, \mu_{2n-2}] + \sum_{k=n}^{2n-1} \pm \mu_k D_k, \quad (3.1,8)$$

Where, the D_k are n-rowed minors of $[\mu_0, \mu_1, \dots, \mu_{2n}]$ not containing μ_{2n} . Similarly

$$[\mu_1, \mu_2, \dots, \mu_{2n+1}] = \mu_{2n+1} [\mu_1, \mu_2, \dots, \mu_{2n-1}] + \sum_{k=n+1}^{2n} \pm \mu_k D'_k \quad (3.1,9)$$

The D'_k are again n-rowed minors of $[\mu_1, \mu_2, \dots, \mu_{2n+1}]$ not containing μ_{2n+1} .

Let, $k \leq m-1$ we have given that-

$$[\mu_0, \mu_1, \dots, \mu_{2k}] \geq 1, \quad [\mu_1, \mu_2, \dots, \mu_{2k+1}] \geq 1 \quad (3.1,10)$$

To prove the same inequalities for $k = m$, using (3.1,7) we observe that the sequence $\{\mu_n\}_0^\infty$ is an increasing function and thus, unity is greater than then no element of the sequence. So,

$$\mu_n > 2 \left(\frac{n}{2}\right)^{\frac{n+4}{4}} (\mu_{n-1})^{\frac{n+2}{2}} \quad (n = 2, 3, \dots).$$

Particularly-

$$\begin{aligned} \mu_{2n} &> 2n^{\frac{n+4}{4}} (\mu_{2n-1})^{n+1} \geq 1 + n^{\frac{n+2}{2}} (\mu_{2n-1})^{n+1} \quad (n = 2, 3, \dots) \\ \mu_{2n+1} &> 2n^{\frac{n+4}{4}} (\mu_{2n})^{n+1} \geq 1 + n^{\frac{n+2}{2}} (\mu_{2n})^{n+1} \quad (n = 2, 3, \dots) \end{aligned} \quad (3.1,11)$$

The elements of D_k are smaller than μ_{2m-1} and those of D'_k are also smaller than μ_{2m} where k ranges over the integer mentioned in the summations (3.1, 8) and (3.1,9). Thus, by Hadamard's upper bound for a determinant- we get,

$$\begin{aligned} |D_k| &\leq m^{\frac{m}{2}} (\mu_{2m-1})^m \quad (k = m, m + 1, \dots, 2m - 1) \\ |D'_k| &\leq m^{\frac{m}{2}} (\mu_{2m})^m \quad (k = m + 1, m + 2, \dots, 2m). \end{aligned}$$

Thus;

$$\begin{aligned} \left| \sum_{k=m}^{2m-1} \pm \mu_k D_k \right| &\leq m (\mu_{2m-1}) m^{\frac{m}{2}} (\mu_{2m-1})^m \\ \left| \sum_{k=m}^{2m} \pm \mu_k D'_k \right| &\leq m (\mu_{2m}) m^{\frac{m}{2}} (\mu_{2m})^m, \end{aligned}$$

so that by (6.23) and (6.24)

$$\begin{aligned} [\mu_0, \mu_1, \dots, \mu_{2m}] &\geq \mu_{2m} - m^{\frac{m+2}{2}} (\mu_{2m-1})^{m+1} \\ [\mu_1, \mu_2, \dots, \mu_{2m+1}] &\geq \mu_{2m+1} - m^{\frac{m+2}{2}} (\mu_{2m})^{m+1} \end{aligned}$$

As per our observation by (3.1,11) we concluded that (3.1,10) is established for $k = m$. Through induction (3.1,10) now holds for all k , and by the Stieltjes moment problem similar to the moments (3.1,7) has an increasing solution $\alpha(t)$. Hence, proved!.

3.2 Determining and non-determining solution

Boas showed that any sequence of sufficiently continuous strong growth leads to a Stieltjes problem which has more than one increasing solution by making use of the last outcome[2-6].

Axiom 3.2(a) If

$$\begin{aligned} \lambda_0 &\geq 1 \\ \lambda_2 &\geq (2\lambda_1 + 2)^2 \\ \lambda_n &\geq (n\lambda_{n-1})^n \quad (n = 1, 3, 4, 5, \dots) \\ \mu_n &= \lambda_{2n} \quad (n = 0, 1, 2, \dots), \end{aligned}$$

So, then there are at least two necessary and significant different increasing functions $\alpha(t)$ such as-

$$\mu_n = \int_0^\infty t^n d\alpha(t) \quad (n = 0, 1, 2, \dots). \quad (3.2,12)$$

There exists a function $\beta(t)$ which is non-negative, increasing, such as given below;

$$\lambda_n = \int_0^\infty t^n d\beta(t^{1/2}) \quad (n = 0, 1, \dots),$$

Thus,

$$\mu_n = \int_0^\infty t^n d\beta(t^{1/2}) \quad (n = 0, 1, 2, \dots).$$

Next, we suppose $\{v_n\}_0^\infty$ be a sequence which is corresponding to the sequence $\{\lambda_n\}_0^\infty$ with an exception $v_1 = \lambda_1 + 1$.

$$v_n \geq (nv_{n-1})^n \quad (n = 1, 2, 3, \dots).$$

Obviously, if n is neither 1 nor 2. Then;

$$\lambda_1 + 1 = v_1 \geq v_0 = \lambda_0$$

$$v_2 = \lambda_2 \geq (2v_1)^2 = (2\lambda_1 + 2)^2.$$

According to Axiom 6.3(a) there is a non-negative increasing function $\gamma(t)$ such as-

$$v_n = \int_0^\infty t^n d\gamma(t) \quad (n = 0, 1, 2, \dots)$$

$$\mu_n = \int_0^\infty t^n d\gamma(t^{1/2}) \quad (n = 0, 1, 2, \dots)$$

Clearly $\beta(t^{1/2})$ is significantly different from $\gamma(t^{1/2})$, for otherwise we should have

$$\int_0^\infty t d\beta(t) = \int_0^\infty t d\gamma(t),$$

which is not possible since, v_1 and λ_1 are unequal.

V. CONCLUSION

Any sequence which increases sufficiently rapidly leads to a soluble Stieltjes problem [with increasing $\alpha(t)$] by a slight modification of the method. The Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval. As every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution. condition for solving moment, determining and non-determining solution.

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