

# L(2,1) and L(3,2,1) Labelling of Certain Star and Wheel Related Graphs

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**ABSTRACT:** An  $L(2,1)$  and  $L(3,2,1)$ -labelling of a graph  $G$  is a function from the vertex set  $V(G)$  to the set of all non-negative integers such that  $d(x, y) + |f(x) - f(y)| \geq 1 + k$ , for  $k = 2$  and  $k = 3$  respectively. The  $L(2,1)$  and  $L(3,2,1)$  labelling number of  $G$ , denoted by  $\square_{2,1}(G)$  and  $\square_{3,2,1}(G)$  respectively which is the smallest number  $k$  such that there is an  $L(2,1)$  and  $L(3,2,1)$  labelling with maximum label  $k$ . In this paper,  $L(2,1)$  and  $L(3,2,1)$ -labelling number of certain star and wheel related graphs were determined.

**KEYWORDS:**  $L(2,1)$  labelling,  $L(3,2,1)$  labelling,  $L(2,1)$  labelling number,  $L(3,2,1)$  labelling number,  $WS(n, r)$  graph,  $S(n, r)$  graph and  $SW(n, r)$  graph.

## I. INTRODUCTION

The frequency assignment problem in wireless communication is that of assigning a frequency to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. The main challenge lies in the fact that the channel allocation problem is NP-complete. Co-channel interference can be caused by many factors from weather conditions to administrative and design issues. This problem is controlled by various radio resource management schemes such as transmit power, user allocation, beamforming, data rates, handover criteria, modulation scheme, error coding scheme, etc. The objective is to utilize the limited radio-frequency spectrum resources and radio network infrastructure as efficiently as possible.

In 1992, Griggs proposed a variation of the channel assignment problem in which “close” transmitters must receive different channels and “very close” transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are “very close” if they are adjacent and “close” if they are of distance two in the graph. The formal graph theoretical definition of  $L(2,1)$  and  $L(3,2,1)$  labelling are as follows: An  $L(2,1)$  and  $L(3,2,1)$  -labelling of a graph  $G$  is a function from the vertex set  $V(G)$  to the set of all non-negative integers such that  $d(x, y) + |f(x) - f(y)| \geq 1 + k$ , for  $k = 2$  and  $k = 3$  respectively. The  $L(2,1)$  and  $L(3,2,1)$  labelling number of  $G$ , denoted by  $\square_{2,1}(G)$  and  $\square_{3,2,1}(G)$  respectively which is the smallest number  $k$  such that there is an  $L(2,1)$  and  $L(3,2,1)$  labelling with maximum label  $k$ .

In the last two decades plenty of research articles in  $L(2,1)$  and  $L(3,2,1)$  were published. In 1999 Fotakis et.al. [7] proved that even for general graphs with diameter 2, the radio coloring problem is NP-hard. Zehui Shao et al. [10] obtained the  $L[2,1]$  labeling of strong product of paths and cycles. Satyabrata Paul et al. [7] proved that, for a circular-arc graph  $G$ ,  $\square_{2,1}(G) \geq \Delta + 3\omega$ , where  $\Delta$  is the maximum degree and  $\omega$  is the clique number. Byeong Moon Kim et al. [3] investigated the  $L(3,2,1)$  labeling for the product of a complete graph and a cycle. S.K.Amanathulla et.al. [1] studied  $L(3,2,1)$  labelling on interval graphs. Bodlaender et al.[2] proved that  $\square_{2,1}(G) \leq 5\Delta - 2\lambda_{2,1}(G) \leq 5\Delta - 2$ , when  $G$  is a permutation graph. Calamoneri et al. [4] completely determined the  $L(2, 1)$ -Labeling of unigraphs, In this paper, we have determined the  $L(2,1)$  and  $L(3,2,1)$ -labelling number of  $n$  -wheel-star graph,  $n$  -star graph and star-wheel graph.

## II. L(2,1) AND L(3,2,1) OF $n$ - WHEEL STAR GRAPH

In this section, we have defined and completely determined the bounds of  $L(2,1)$  and  $L(3,2,1)$  labelling for  $n$  -wheel Star graph.

**Definition 2.1:** Let  $u_i, 1 \leq i \leq n$  be the vertices of the star graph  $S_{n+1}$  with hub at  $w$  and let  $v_i, 1 \leq i \leq nr$  be the vertices in the wheel graphs with hubs  $u_i, i = 1, 2 \dots n$ . The graph obtained is called  $n$ -wheel star graph and it is denoted by  $WS(n, r)$ . (See Fig. 1).

The number of vertices and edges in  $WS(n, r)$  are  $nr + n + 1$  and  $nr + n$  respectively. Its diameter and radius are 4 and 2 respectively.

**Theorem 2.1:** Let  $G$  be a  $n$ -wheel star graph  $WS(n, r)$ . Then  $L(2,1)$  labelling number of  $G$  satisfies  $\chi_{2,1}(G) \leq 2r + n - 1$ .

**Proof:** Define a mapping  $f : V(G) \rightarrow N \cup \{0\}$  by  
 $f(v_{j_r+i}) = 2i - 1, i = 1, 2 \dots r, j = 0, 1 \dots n - 1$   
 $f(u_i) = 2r + i, i = 1, 2 \dots n,$   
 $f(w) = 0.$  (See Fig. 1 (a)).

Next we must verify that,  $d(x, y) + |f(x) - f(y)| \geq 3$  for any pair of vertices  $x$  and  $y$  in  $WS(n, r)$ .

**Case 1:** Let  $x$  and  $y$  be any two vertices in the wheels.

**Case 1.1:** If  $x$  and  $y$  lies in the same wheel, then  $x = v_{(k-1)r+m}$  and  $y = v_{(k-1)r+s}, m \neq s, 1 \leq k \leq r$ . Therefore, the function values of  $x$  and  $y$  are  $2m - 1$  and  $2s - 1$  respectively. Also  $d(x, y) \geq 1$ . Hence the  $L(2,1)$  - labelling condition becomes  $d(x, y) + |f(x) - f(y)| \geq 1 + |2m - 2s| \geq 3$ , since  $m \neq s$ .

**Case 1.2:** Suppose  $x$  and  $y$  lies in the different wheels, then  $x$  and  $y$  are of the form  $v_{(k-1)r+s}$  and  $v_{(m-1)r+t}, 1 \leq k \neq m \leq r$ . Hence  $f(x) = 2s - 1, f(y) = 2t - 1$  and the distance between  $x$  and  $y$  is exactly 4, which implies that  $d(x, y) + |f(x) - f(y)| = 4 + |2(s - t)| \geq 4$ .

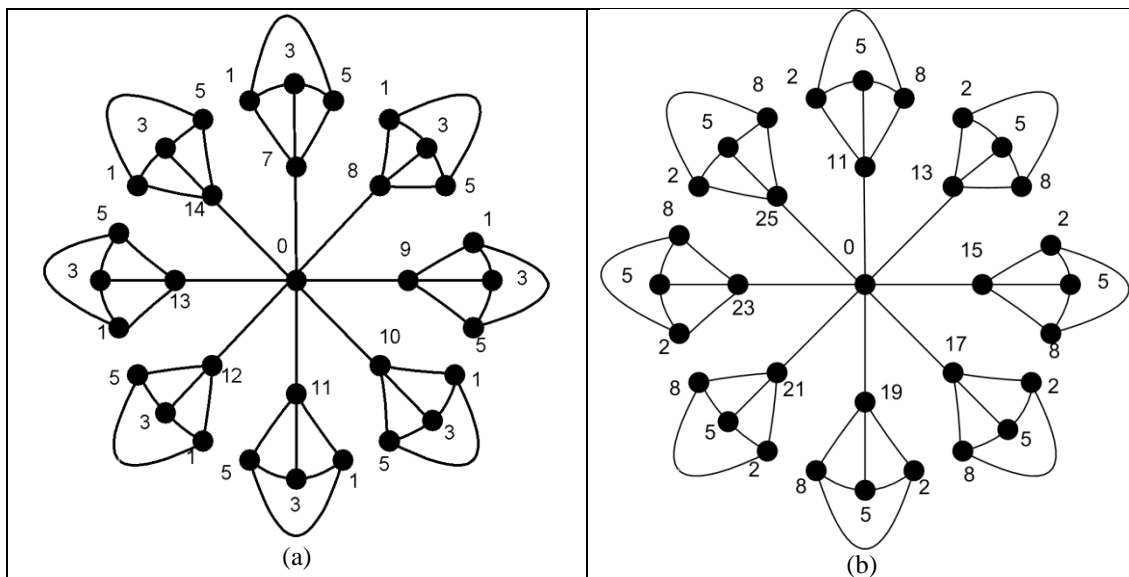


Fig.1: L(2,1) and L(3,2,1) labelling of  $WS(n, r)$  with  $n = 8, r = 3$ .

**Case 2:** Suppose  $x$  is a vertex in the star graph  $S_{n+1}$ , other than the center vertex and  $y$  is any vertex in the wheel, then  $x = u_m$  and  $y = v_{(k-1)r+s}$ . Therefore,  $f(x) = 2r + m, f(y) = 2s - 1, 1 \leq m \leq n, 1 \leq s \leq r$ . Since the distance between  $x$  and  $y$  is at least 1, the  $L(2,1)$  labelling condition satisfies:  $d(x, y) + |f(x) - f(y)| \geq 1 + |2r + m - 2s - 1| \geq 1 + 2r - s + m + 1 \geq 3$ .

**Case 3:** Suppose  $x$  and  $y$  are any two vertices in the star graph  $S_{n+1}$  other than the center vertex, then  $x$  and  $y$  are of the form  $x = u_l$  and  $x = u_m, l \neq m$ . Therefore  $f(x) = 2r + l, f(y) = 2r + m$  and  $d(x, y) = 2$ . Hence  $|f(x) - f(y)| \geq |l - m| > 1$ , since  $l \neq m$ , which forces that  $d(x, y) + |f(x) - f(y)| \geq 3$ .

**Case 4:** Suppose  $x$  is the center vertex of the star graph and  $y$  is any other vertex in the star graph  $S_{n+1}$ , then  $f(x) = 0, f(y) = 2r + 1, 1 \leq m \leq n$  and  $d(x, y) \geq 1$ . Therefore,  $d(x, y) + |f(x) - f(y)| \geq 1 + |2r + m - 0| \geq 3$ .

**Case 5:** Suppose  $x = w$  and  $y$  lie in any wheel, then  $f(w) = 0$  and  $f(y) = 2m - 1, 1 \leq m \leq r$ . Also the distance between  $x$  and  $y$  is exactly 2. Hence,  $d(x, y) + |f(x) - f(y)| \geq 2 + |2m - 1| \geq 3$ . Thus,  $f$  is a valid  $L(2,1)$  labelling. Therefore,  $\lambda_{2,1}(G) \leq 2r + n - 1$ .

**Theorem 2.2:** The  $L(3,2,1)$  labelling number of  $n$ -wheel star graph satisfies,  $\lambda_{3,2,1}(WS(n, r)) \leq 3r + 2n$ .

**Proof:** The vertices in  $WS(n, r)$  are labeled by the mapping  $f(v_{jr+i}) = 3i - 1, i = 1, 2 \dots r, j = 0, 1 \dots n - 1, f(u_i) = 3r + 2i, i = 1, 2 \dots n$  and  $f(w) = 0$ . (See Fig.1(b)).

Next we verify  $f$  satisfies the  $L(3,2,1)$  labelling condition,  $d(x, y) + |f(x) - f(y)| \geq 4 \forall x, y \in V(WS(n, r))$ .

**Case 1:** Let  $x$  and  $y$  be any two vertices in the wheel.

**Case 1.1:** If  $x$  and  $y$  lies in the same wheel, then  $x = v_{(k-1)r+m}$  and  $y = v_{(k-1)r+s}, m \neq s, 1 \leq k \leq r$ . Therefore,  $f(x) = 3m - 1, f(y) = 3s - 1$  and  $d(x, y) \geq 1$ . Hence,  $d(x, y) + |f(x) - f(y)| \geq 1 + |3m - 3s| \geq 4$ , since  $m \neq s$ .

**Case 1.2:** Suppose  $x$  and  $y$  lies in the different wheel graphs, then  $f(x) = 3s - 1, f(y) = 3t - 1, 1 \leq k \neq m \leq r$ . Also distance between  $x$  and  $y$  is exactly 4. Therefore,  $d(x, y) + |f(x) - f(y)| = 4 + |3(s - t)| \geq 4$ .

**Case 2:** Suppose  $x$  is a vertex in the star graph  $S$  and  $y$  is any vertex in the wheel, then  $x = u_m$  and  $y = v_{(k-1)r+s}$ . Therefore,  $f(x) = 3r + 2m, f(y) = 3s - 1, 1 \leq m \leq n, 1 \leq s \leq r$ . Since the distance between  $x$  and  $y$  is at least 1, we have,  $d(x, y) + |f(x) - f(y)| \geq 1 + |3r + 2m - (3s - 1)| \geq 1 + |3(r - s) + 2m + 1| \geq 4$ .

**Case 3:** Suppose  $x$  and  $y$  are any two vertices in the star graph, that is,  $x = u_l$  and  $x = u_m, l \neq m$ , then  $f(x) = 3r + 2l$  and  $f(y) = 3r + 2m$ . Also  $d(x, y) = 2$  and  $|f(x) - f(y)| \geq |2(l - m)| > 2$ , since  $l \neq m$ . Hence  $d(x, y) + |f(x) - f(y)| \geq 4$ .

**Case 4:** Suppose  $x$  lie in the star  $S$  and  $y = w$ , then  $f(x) = 3r + 2m, 1 \leq m \leq n, f(y) = 0$  and  $d(x, y) \geq 1$ . Therefore,  $d(x, y) + |f(x) - f(y)| \geq 1 + |3r + 2m - 0| \geq 4$ .

**Case 5:** Suppose  $x = w$  and  $y$  lie in any wheel, then  $f(w) = 0$  and  $f(y) = 3m - 1, 1 \leq m \leq r$ . Also the distance between  $x$  and  $y$  is exactly 2. Hence,  $d(x, y) + |f(x) - f(y)| \geq 2 + |3m - 1| \geq 3$ .

Thus the  $L(3,2,1)$  labelling condition is satisfied for every pair of vertices in  $WS(n, r)$ .

Hence,  $\lambda_{3,2,1}(G) \leq r + n + 3$ .

### III. L(2,1) AND L(3,2,1) LABELLING OF $n$ -STAR GRAPH

In this section, the upper bound of  $L(2,1)$  and  $L(3,2,1)$  labelling for  $n$ -Star graph has been obtained.

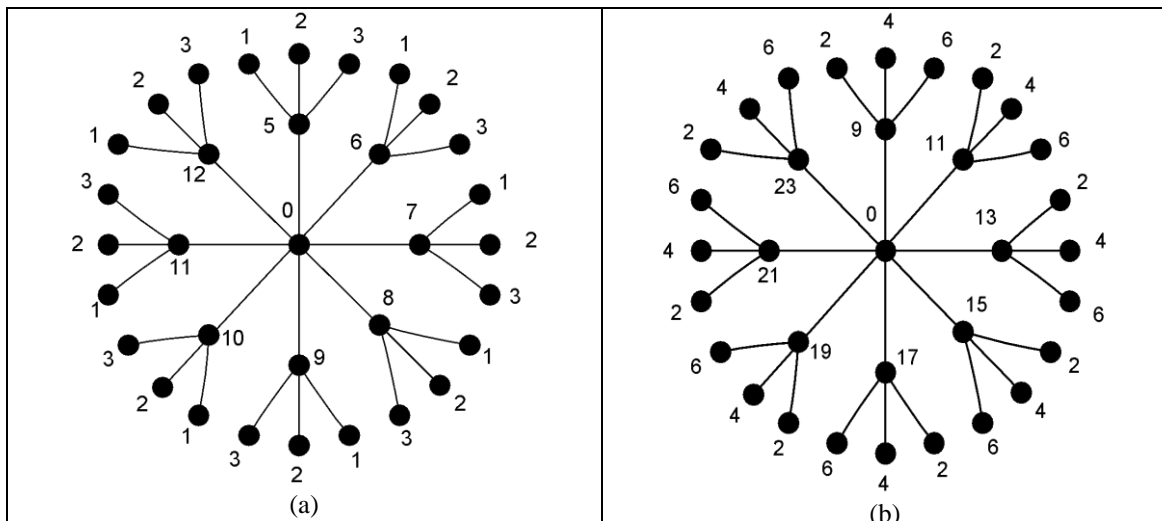


Fig. 2: A  $L(2,1)$  and  $L(3,2,1)$  labelling of  $S(n, r)$  with  $n = 8$  and  $r = 3$  which attains the bound.

**Definition 3.1:** Let  $u_i, 1 \leq i \leq n$  be the vertices of the inner star graph  $S_{n+1}$  with hub at  $w$  and let  $v_i, 1 \leq i \leq nr$  be the vertices in the  $n$  –outer star graphs with hubs  $u_i, i = 1,2 \dots n$ . The graph obtained is called the  **$n$ -Star graph** and it is denoted by  $S(n, r)$ . (See Fig 2). The diameter of the graph is 4.

**Theorem 3.1:** Let  $G$  be a  $n$ -star graph  $S(n, r)$ . Then  $L(2,1)$  labelling number of  $G$  satisfies  $\lambda_{2,1}(G) \leq r + n + 1, r \geq 2$ .

**Proof:** Define a mapping  $f : V(G) \rightarrow N \cup \{0\}$  by  
 $f(v_{jr+i}) = i, i = 1,2 \dots r, j = 0,1 \dots n - 1,$   
 $f(u_i) = r + i + 1, i = 1,2 \dots n.$   
 $f(w) = 0.$  (See Fig 3.1 (a)).

As the proof is similar to the Theorem 2.1, we omit the rest of the proof.

**Theorem 3.2:** The  $L(3,2,1)$  labelling number of  $n$ -star graph  $S(n, r)$  satisfies  $\lambda_{3,2,1}(G) \leq 2r + 2n + 1$ .

**Proof:** The vertices in  $S(n, r)$  are labelled as follows:  
 $f(v_{jr+i}) = 2i, i = 1,2 \dots r, j = 0,1 \dots n - 1,$   
 $f(u_i) = 2r + 2i + 1, i = 1,2 \dots n,$   
 $f(w) = 0.$  (See Fig. 4.1).

The rest of the proof is similar to Theorem 2.2.

#### IV. L(2,1) AND L(3,2,1) LABELLING OF $n$ –STAR-WHEEL GRAPH

In this section, we have defined and determined the upper bound of  $L(2,1)$  and  $L(3,2,1)$  labelling for star wheel graph.

**Definition 4.1:** Let  $u_i, 1 \leq i \leq n$  be the vertices of the wheel  $W_{n+1}$  with  $w$  as its center vertex. let  $S_r^i, 1 \leq i \leq n$  be the  $n$  number of star graphs with hubs at  $u_i$  respectively. The graph constructed is called  **$n$  –star-wheel graph** and is denoted by  $SW(n, r)$ . (See Fig. 3).

The number of vertices and edges are  $nr + n$  and  $nr + 2n$  respectively. Its diameter is 4.

**Theorem 4.1:** Let  $G$  be  $n$ -star-wheel graph  $SW(n, r)$ . Then  $L(2,1)$  labelling number of  $G$  satisfies  $\lambda_{2,1}(G) \leq r + n + 1$ .

**Proof:** First we have name the vertices of the star graphs  $S_r^i, 1 \leq i \leq n$  as  $v_1, v_2 \dots v_r \dots v_{2r} \dots v_{3r} \dots v_{nr}$ . Next we define a mapping  $f : V(G) \rightarrow N \cup \{0\}$  as follows:

$f(v_{jr+i}) = i, i = 1,2 \dots r, j = 0,1 \dots n - 1$   
 $f(u_i) = \{r + \lceil \frac{i}{2} \rceil + 1, i \text{ is odd } r + \frac{i}{2} + 1, i \text{ is even } , i = 1,2 \dots n$   
 $f(w) = 0.$  (See Fig. 3 (a)).

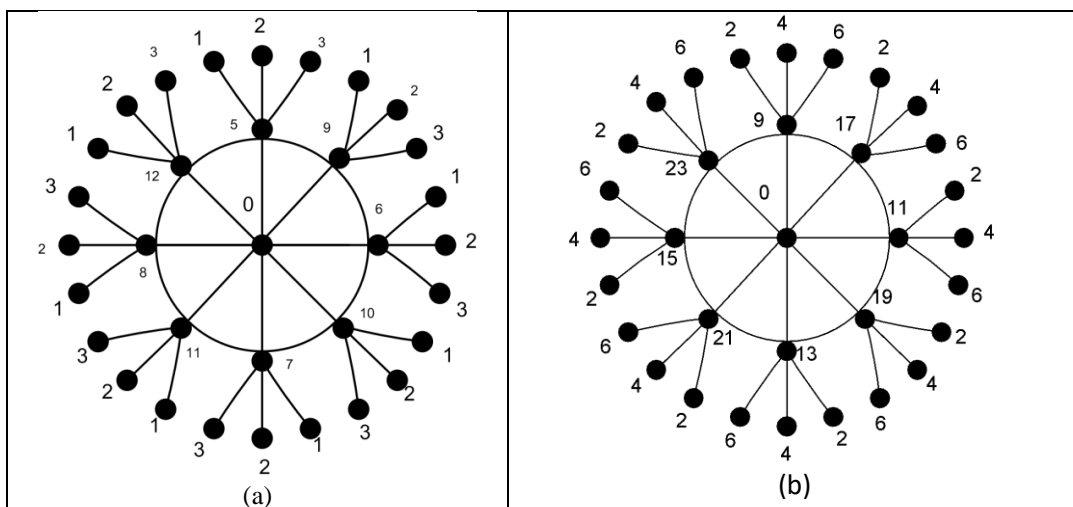


Fig. 3: A graph  $SW(8,3)$  and its  $L(2,1)$  and  $L(3,2,1)$  labelling which attains the bound

As in Theorem 2.1, we can easily verify the  $L(2,1)$  labelling condition for the  $n$ -star-wheel graph.

Hence  $\lambda_{2,1}(SW(n, r)) \leq r + n + 1$ .

**Theorem 4.2:** Let  $G$  be  $n$ -star-wheel graph  $SW(n, r)$ . Then  $L(3, 2, 1)$  labelling number of  $G$  satisfies  $\lambda_{3,2,1}(G) \leq 2r + 2n + 1$ .

**Proof:** We define a mapping  $f: V(G) \rightarrow N \cup \{0\}$  as follows:

$$f(v_{jr+i}) = 2i, \quad i = 1, 2 \dots r, \quad j = 0, 1 \dots n$$

$$f(u_{2i-1}) = 2r + 2i + 1, \quad i = 1, 2 \dots \lfloor \frac{n}{2} \rfloor$$

$$f(u_{2i}) = 2r + 2\lfloor \frac{n}{2} \rfloor + 2i + 1, \quad i = 1, 2 \dots \lfloor \frac{n}{2} \rfloor$$

$$f(w) = 0. \text{ (See Fig. 3).}$$

The rest of the proof is left to the reader.

## V. CONCLUSION

Efficient allocation of channels for wireless communication in different network scenarios has become an extremely important topic of recent research in graph labelling problems. In this paper we have determined the  $L(2,1)$  and  $L(3,2,1)$  labelling for certain star and wheel related graphs. This work can be further extended to other types of interconnection networks such as Oxide network, Silicate networks, butterfly networks etc.

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