

# Inclusive Lucky Labeling for Some Special Families of Graphs

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**ABSTRACT:** In this paper Inclusive lucky labeling of some special types of graphs like Crown graph, Helm graph, Split graph, Book graph, Amal graph, Gear graph are discussed and the Inclusive lucky number of these graphs are found.

**KEYWORDS:** Inclusive lucky labelling, Inclusive lucky number, Crown graph, Helm graph, Split graph, Book graph, Amal graph, Gear graph.

## I. INTRODUCTION

All the graphs considered are simple and undirected. For the basic definitions of graph theory refer Harary [3] and Douglas B. West[2]. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Rosa [1] in the year 1967 introduced the concept of labelling. The concept of labeling has much importance in graph theory as it is being used in various fields such as communication networks, coding theory, astronomy etc. In this paper the Inclusive lucky labeling of Crown graph, Helm graph, Split graph, Book graph, Amal graph, Gear graph are discussed and the Inclusive lucky numbers are found for these graphs.

First, some basic definitions for the study of this paper are discussed in this section.

### Definition 1.1

Let  $G$  be a Simple Graph with Vertex set  $V(G)$  and Edge set  $E(G)$  respectively. Let  $v \in V(G)$ . The open neighborhood  $N(v)$  of a vertex  $v$  in a graph  $G$  is the set of all vertices adjacent to  $v$  in  $G$ .  $N[v] = N(v) \cup \{v\}$  is called the closed neighborhood of  $v$ .

### Definition 1.2

Let  $G$  be a Simple Graph with vertex set  $V(G)$  and edge set  $E(G)$  respectively. Let  $f : V(G) \rightarrow N$  be a labeling of the vertices of a graph  $G$  by positive integers. Define  $S(v) = \sum_{u \in N(v)} f(u)$ , as the sum of neighborhood of vertex  $v$ , where  $N(v)$  denotes the open neighborhood of  $v \in V$ . A labeling  $f$  is lucky if  $S(u) \neq S(v)$  for every pair of adjacent vertices  $u$  and  $v$ . The lucky number of a graph  $G$ , denoted by  $\eta(G)$ , is the least positive integer  $k$  such that  $G$  has a lucky labeling with  $\{1, 2, \dots, k\}$  as the set of labels.

A graph which admits lucky labeling is the lucky labeled graph.

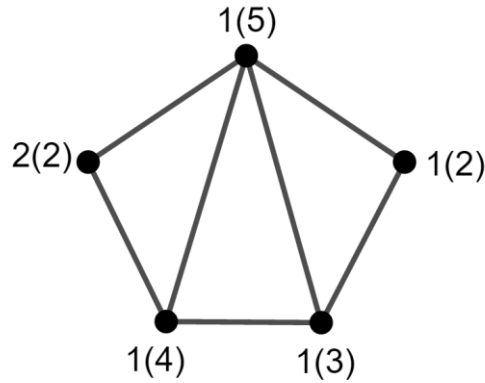


Figure 1.1: Lucky Labeling

**Definition 1.3**

Let  $G$  be a Simple Graph with Vertex set  $V(G)$  and Edge set  $E(G)$  respectively. Let  $f : V(G) \rightarrow N$  be a labeling of the vertices of a graph  $G$  by positive integers. Define  $S(v) = \sum_{u \in N[v]} f(u)$ , as the sum of neighborhood of vertex  $v$ , where  $N[v]$  denotes the closed neighborhood of  $v \in V$ . A labeling  $f$  is an Inclusive lucky labeling if  $S(u) \neq S(v)$  for every pair of adjacent vertices  $u$  and  $v$ . The Inclusive lucky number of a graph  $G$ , denoted by  $\eta_l(G)$ , is the least positive integer  $k$  such that  $G$  has an Inclusive lucky labeling with  $\{1, 2, \dots, k\}$  as the set of labels.

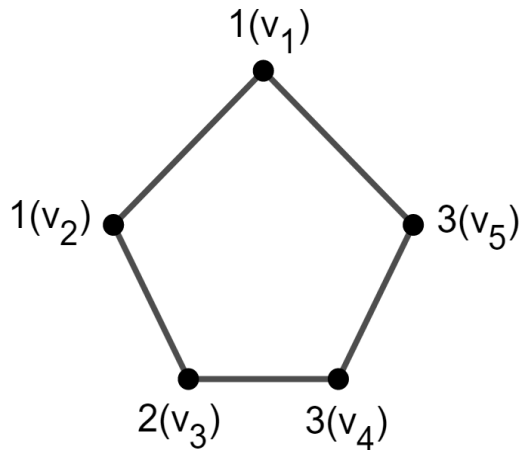


Figure 1.2: Inclusive Lucky Labeling

**Definition 1.4**

The wheel graph denoted by  $W_n$  is a graph with  $n$  vertices ( $n > 4$ ), formed by connecting a single vertex to all vertices of an  $(n - 1)$  cycle.

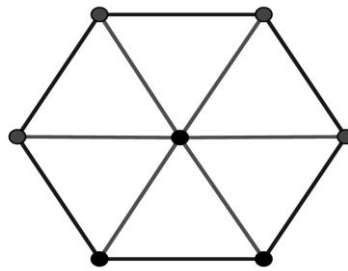


Figure 1.3: Wheel,  $W_7$

**Definition 1.5**

The Crown graph is obtained from a graph  $G$  by attaching a pendant vertex from each vertex of the cycle graph  $C_n$  and this graph is denoted by  $Cr_n$ .

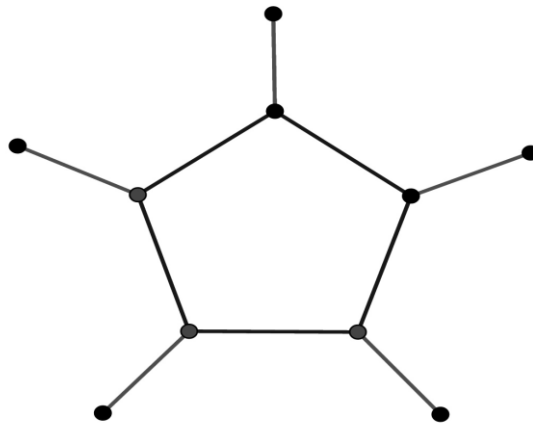


Figure 1.4: Crown,  $Cr_5$

**Definition 1.6**

The Helm graph  $H_n$  is obtained from a wheel graph  $W_n$  by attaching a pendant edge at each vertex of the  $(n - 1)$  cycle.

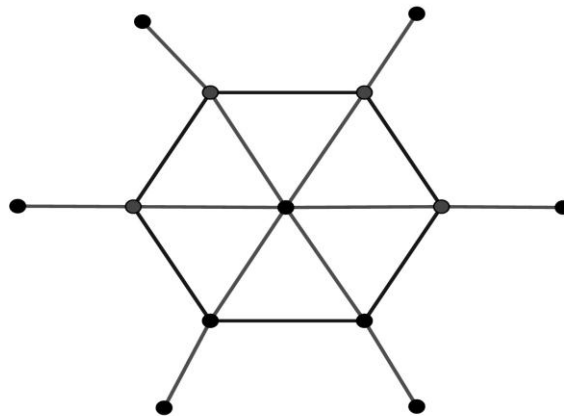


Figure 1.5: Helm,  $H_7$

**Definition 1.7**

The Gear graph  $G_r$  also known as a bipartite wheel graph, is a wheel graph  $W_{n+1}$  with a vertex added between each pair of adjacent vertices of the outer cycle. Gear graph  $G$  has  $2n + 1$  vertices and  $3n$  edges.

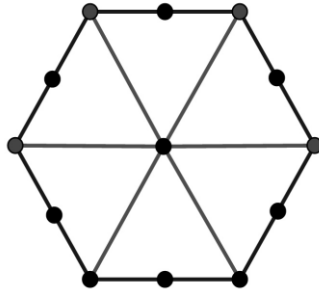


Figure 1.6: Gear,  $Gr_6$

**Definition 1.8**

The Book graph  $B_n$  is a graph obtained from One edge union of cycles of same length. The common edge is called the base of the book. If we consider  $t$  copies of cycles of length  $n \geq 3$ , the book is denoted by  $B_n^t$ . If  $n = 4$ , then the book  $B$  is called book with rectangular (or quadrilateral).

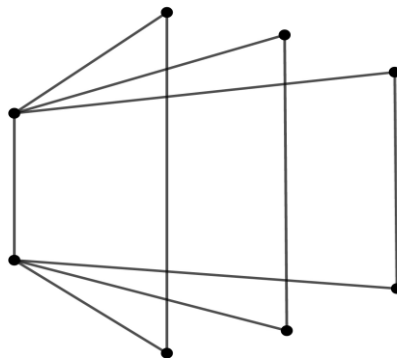


Figure 1.7: Book,  $B_3$

**Definition 1.9**

The split graph is a graph whose vertices can be partitioned into two sets  $V_1$  and  $V_2$ , where the vertices  $V_1$  form a complete graph and the vertices in  $V_2$  are independent.

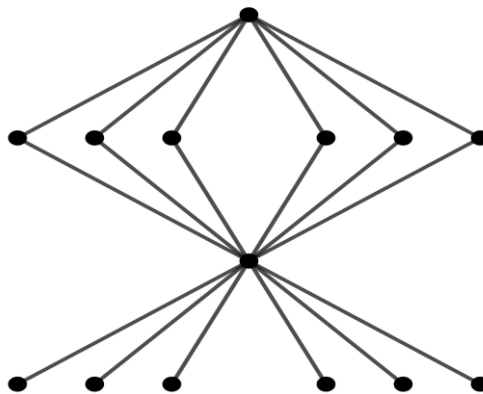


Figure 1.8: Split,  $SP_{1,6}$

**Definition 1.10**

The Amal graph is a graph obtained from two graphs. Let  $G$  and  $H$  be two graphs. Let  $u \in V(G)$  and

$v \in V(H)$ . Then the amalgamation of  $(G, u)$  with  $(H, v)$  is the graph obtained by forming the disjoint union of  $G$  and  $H$  and then identifying  $u$  and  $v$ . It is denoted as  $\text{Amal}(G, H, (u, v))$ .

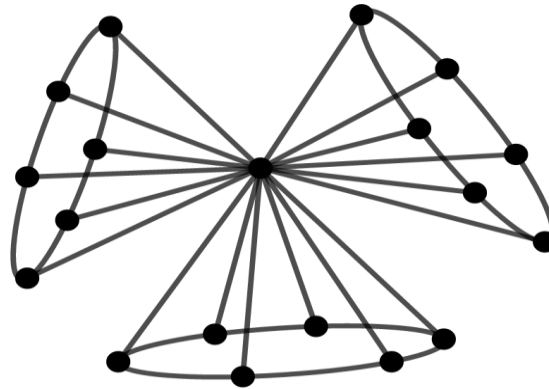


Figure 1.9: Amal,  $A_6$

## II. MAIN RESULTS

### Theorem 2.1:-

The inclusive lucky number of crown graph  $Cr_n$  when  $n > 3$  is given by  $\eta_i(Cr_n) = \begin{cases} 2 & \text{when } n \text{ is even} \\ 3 & \text{when } n \text{ is odd} \end{cases}$

#### Proof:

Let  $G$  be a Simple Crown Graph with Vertex set  $V(G)$ , Edge set  $E(G)$  and  $|V(G)| = 2n$  respectively.

#### Case (i): When $n$ is even.

Let the vertices on the cycle be  $v_1, v_2, v_3, v_4, \dots, v_n$  and let the pendant vertex attached to each vertex of the cycle be  $u_1, u_2, u_3, \dots, u_n$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$ . Label the vertices of the cycle  $v_i$  ( $i = 1, 2, \dots, n$ ) by the labels  $\{1, 2\}$  alternatively and label the pendant vertices  $u_i$  ( $i = 1, 2, \dots, n$ ) by the labels  $\{1, 2\}$  in such a way that if the vertex  $v_i$  is labeled  $\{1\}$  then label vertex  $u_i$  with  $\{2\}$  or if the vertex  $v_i$  is labeled  $\{2\}$  then label the vertex  $u_i$  with  $\{1\}$ . Then,  $S(v)$  is  $\{5, 7\}$  and  $S(u)$  is  $\{3\}$ .

This induces an inclusive lucky labeling of  $Cr_n$  and  $\eta_i(Cr_n) = 2$  when  $n$  is even.

#### Case (ii): When $n$ is odd.

Let the vertices on the cycle be  $v_1, v_2, v_3, v_4, \dots, v_n$  and let the pendant vertex attached to each vertex of the cycle be  $u_1, u_2, u_3, \dots, u_n$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$ . Label the vertices of the cycle  $v_i$  ( $i = 1, 2, \dots, n$ ) by the labels  $\{1, 2, 3\}$  in such a way that  $\{1\}$  is labeled 2-times and  $\{2\}$  and  $\{3\}$  is labeled alternatively. Label the pendant vertices  $u_i$  ( $i = 1, 2, \dots, n$ ) in such a way that if the vertex  $v_i$  is labeled  $\{1\}$  then label vertex  $u_i$  with  $\{2\}$  or if the vertex  $v_i$  is labeled  $\{2\}$  then label the vertex  $u_i$  with  $\{1\}$ . Label the first vertex  $v_1$  with label  $\{1\}$  and its pendent vertex with label  $\{3\}$ . Then,  $S(v)$  is  $\{3, 4, 5, 6, 7\}$  and  $S(u)$  is  $\{3, 4\}$ .

This induces an inclusive lucky labeling of  $Cr_n$  and  $\eta_i(Cr_n) = 3$  when  $n$  is odd.

### Theorem 2.2:-

The inclusive lucky number of Helm graph  $H_n$  when  $n \geq 5$  is given by  $\eta_i(H_n) = \begin{cases} 3 & \text{when } n \text{ is even} \\ 2 & \text{when } n \text{ is odd} \end{cases}$

#### Proof:

#### Case(i): When $n$ is odd

Let the single vertex at the center be  $v_1$  and the other vertices of  $(n - 1)$  cycle that are joined to the vertex  $v_1$  are  $v_2, v_3, v_4, \dots, v_n$ . Let the pendant vertex attached to each vertices of  $(n - 1)$  cycle be  $u_1, u_2, u_3, \dots, u_{n-1}$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, \dots, v_n, u_1, u_2, u_3, \dots, u_{n-1}\}$ . Label the single vertex at the center  $v_1$  with  $\{1\}$  and label the other vertices of  $(n - 1)$  cycle  $v_i$  ( $i = 2, 3, \dots, n$ ) by the label  $\{1\}$  and  $\{2\}$  alternatively. Label the vertices of the pendant vertex  $u_i$  ( $i = 1, 2, \dots, n - 1$ ) by the label  $\{1\}$  and  $\{2\}$  such a way that if the vertex  $v_i$  is labeled  $\{1\}$  then label vertex  $u_i$  with  $\{2\}$  or if the vertex  $v_i$  is labeled  $\{2\}$  then label vertex  $u_i$  with  $\{1\}$ . Then,  $S(v_1)$  is sum of all

vertex labels except the pendant vertices,  $S(v_i)$  ( $i = 2, 3, \dots, n$ ) is  $\{6, 8\}$  and  $S(u_i)$  is  $\{3\}$ . This induces an Inclusive lucky labeling of  $H_n$  and  $\eta_i(H_n) = 2$  when  $n$  is odd.

**Case (ii): When  $n$  is even**

**Sub case (i): If  $n = 6$**

Let the single vertex at the center be  $v_1$  and the other vertices of  $(n - 1)$  cycle that are joined to the vertex  $v_1$  be  $v_2, v_3, v_4, v_5, v_6$ . Let the pendant vertex attached to each vertices of  $(n - 1)$  cycle be  $u_1, u_2, u_3, u_4, u_5$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, v_5, v_6, u_1, u_2, u_3, u_4, u_5\}$ . Label the vertex at the center  $v_1$  as  $\{1\}$  and label the other vertices of  $(n - 1)$  cycle  $v_i$  ( $i = 2, 3, \dots, 6$ ) by the label  $\{1, 2\}$  in such a way that label  $\{2\}$  once,  $\{1\}$  once,  $\{2\}$  once and label  $\{1\}$  2-times. Label the vertices of the pendant vertex  $u_i$  ( $i = 1, 2, \dots, 5$ ) by the label  $\{1, 2, 3\}$  such a way that  $\{2\}$  one time,  $\{3\}$  one time and  $\{1\}$  three times.

Then,  $S(v_1)$  is sum of all the labels of the vertices except the pendant vertices,  $S(v_i)$  ( $i = 2, 3, \dots, 6$ ) is  $\{6, 7, 8\}$  and  $S(u_i)$  is  $\{3, 4\}$ , this induces an Inclusive lucky labeling of  $H_n$  and  $\eta_i(H_n) = 3$  when  $n = 6$ .

**Sub case (ii): If  $n \geq 8$**

Let the single vertex at the center be  $v_1$  and the other vertices of  $(n - 1)$  cycle that are joined to the vertex  $v_1$  be  $v_2, v_3, v_4, \dots, v_n$ . Let the pendant vertex attached to each vertices of  $(n - 1)$  cycle be  $u_1, u_2, u_3, \dots, u_{n-1}$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, \dots, v_n, u_1, u_2, u_3, \dots, u_{n-1}\}$ . Label the vertex  $v_1$  at the center be with  $\{1\}$  and label the other vertices of  $(n - 1)$  cycle  $v_i$  ( $i = 2, 3, \dots, n$ ) by the label  $\{1\}$  2-times and  $\{2, 1\}$  alternatively. Label the vertices of the pendant vertex  $u_i$  ( $i = 1, 2, \dots, (n - 1)$ ) such a way that the first pendant vertex is labeled  $\{3\}$  and the other vertices  $v_i$  is labeled  $\{1\}$  then label the vertices  $u_i$  with  $\{2\}$  or if  $v_i$  is labeled  $\{2\}$  then label the vertices  $u_i$  with  $\{1\}$ .

then,  $S(v_1)$  is sum of all vertices except the pendant vertices,  $S(v_i)$  ( $i = 2, 3, \dots, n$ ) is  $\{6, 7, 8\}$  and  $S(u_i)$  is  $\{3, 4\}$ . This induces an inclusive lucky labeling of  $H_n$  and  $\eta_i(H_n) = 3$  if  $n \geq 8$ .

Therefore, this induces an Inclusive lucky labeling of  $H_n$  and  $\eta_i(H_n) = 3$  when  $n$  is even.

**Theorem 2.3:-**

The inclusive lucky number of split graph of  $K_{1,n}$  is given by  $\eta_i(K_{1,n}) = 1$  where  $n \geq 3$ .

**Proof:**

Let  $u_1, u_2, u_3, \dots, u_n$  be the vertices of star graph  $K_{1,n}$  with  $v_1$  as an apex vertex. Let  $G$  be the splitting graph of  $K_{1,n}$  and  $v_1', u_1', u_2', u_3', \dots, u_n'$  be the newly added vertices with  $K_{1,n}$  to form  $G$ .

Then the vertex set  $V = \{v_1, u_1, u_2, u_3, \dots, u_n, v_1', u_1', u_2', u_3', \dots, u_n'\}$ . Label all the vertices  $v_1, u_1, u_2, u_3, \dots, u_n, v_1', u_1', u_2', u_3', \dots, u_n'$  with  $\{1\}$ . Then,  $S(v_1)$  is the sum of all the vertex labels of  $u_i$  ( $i = 1, 2, 3, \dots, n$ ) which is equal to  $n+1$ .  $S(v_1')$  is the sum of all the vertex labels of  $u_i$  and  $u_i'$  ( $i = 1, 2, 3, \dots, n$ ) which is equal to  $2n + 1$ .  $S(u_i)$  and  $S(u_i')$  ( $i = 1, 2, 3, \dots, n$ ) is  $\{2, 3\}$ .

This induces an Inclusive lucky labeling of split graph of  $(K_{1,n})$  and  $\eta_i(K_{1,n}) = 1$ .

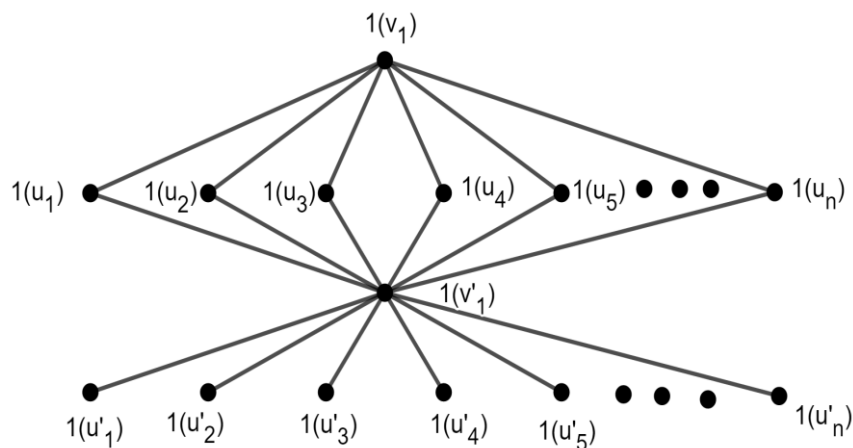


Figure 2.1: Split graph of  $K_{1,n}$

**Theorem 2.4:-**

The inclusive lucky number of book graph  $B_n$  is given by  $\eta_i(B_n) = 2$ .

**Proof:**

Let  $G$  be the Book graph denoted by  $B_n$ . Let the vertices of common edge be  $v_1$  and  $v_2$ . Let the vertices joined to  $v_1$  by the pendant edges be  $u_1, u_2, u_3, \dots, u_n$ . and the vertices joined to  $v_2$  by the pendant edges be  $w_1, w_2, w_3, \dots, w_n$ . Then the vertex set  $V = \{v_1, v_2, u_1, u_2, u_3, \dots, u_n, w_1, w_2, w_3, \dots, w_n\}$ . Label all the vertices  $v_1$  and  $u_i$  ( $i = 1, 2, \dots, n$ ) with label  $\{2\}$  and label all the vertices  $v_2$  and  $w_i$  ( $i = 1, 2, \dots, n$ ) with  $\{1\}$ . Then,  $S(v_1)$  is sum of all the labels of the vertices of  $u_i$  ( $i = 1, 2, \dots, n$ ),  $v_1$  and  $v_2$  which is  $2n + 3$   $S(v_2)$  is sum of all the labels of the vertices of  $w_i$  ( $i = 1, 2, \dots, n$ ),  $v_1$  and  $v_2$  which is  $n + 3$ .

$S(u_i)$  ( $i = 1, 2, \dots, n$ ) is  $\{5\}$  and  $S(w_i)$  ( $i = 1, 2, \dots, n$ ) is  $\{4\}$ .

This induces an Inclusive lucky labeling of  $B_n$  and  $\eta_i(B_n) = 2$ .

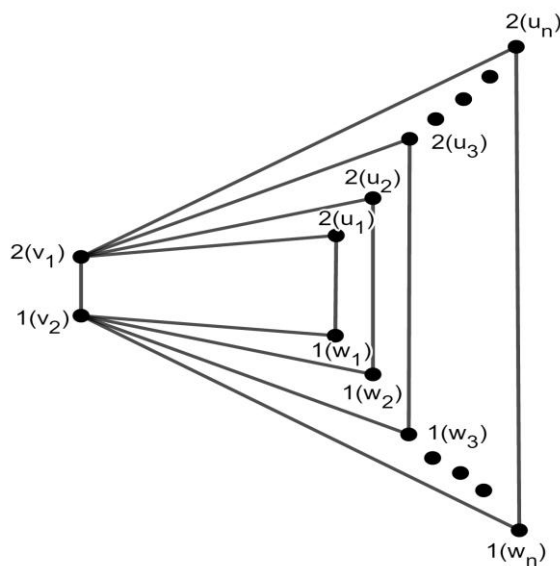


Figure 2.2: Book,  $B_n$

**Theorem 2.5:-**

The inclusive lucky number of Amal graph  $Amal(W_6, x_i | i = 1, 2, 3 \dots)$  is given by  $\eta_i(Amal(W_6, x_i | i = 1, 2, 3))=2$

**Proof:**

Let  $G$  be the Amal graph on  $n$  vertices which is odd. Let the single vertex at the center be  $v_1$  and the other  $(n - 1)$  vertices are made into cycle on 6 vertices that are joined to the vertex  $v_1$  be  $v_2, v_3, v_4, \dots, v_n$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, \dots, v_n\}$ . Label the vertex at the center  $v_1$  as  $\{1\}$  and label the vertices  $v_i$  ( $i = 2, 3, \dots, n$ ) by the label  $\{1, 2\}$  alternatively. Then  $S(v_1)$  is sum of all the labels of the vertices  $S(v_2)$  to  $S(v_n)$  is  $\{5, 6\}$  respectively.

This induces an Inclusive lucky labeling of  $Amal(W_6, x_i | i = 1, 2, 3 \dots)$  and  $\eta_i(Amal(W_6, x_i | i = 1, 2, 3)) = 2$ .

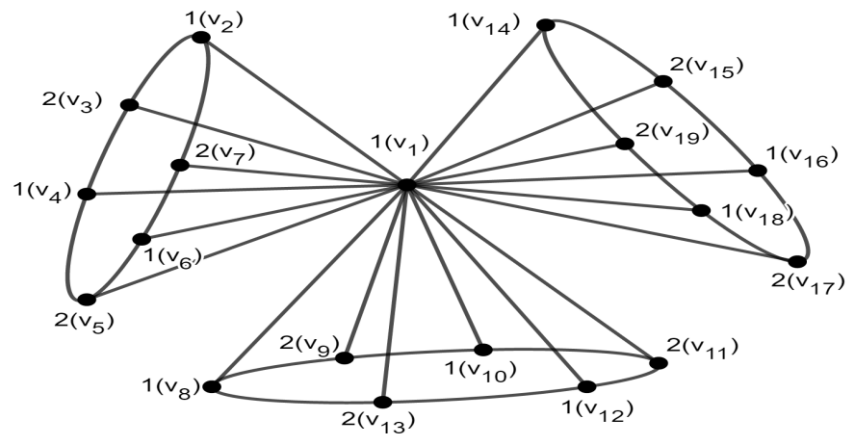


Figure 2.3: Amal Graph

**Theorem 2.6:-**

The inclusive lucky number of Gear graph  $Gr_n$  is given by  $\eta_l(Gr_n) = 1$  where  $n \geq 5$ .

**Proof:**

Let  $G$  be the Gear graph  $Gr_n$  with  $n$  vertices. Let the single vertex at the center be  $v_1$  and the other vertices of outer cycle be  $v_2, v_3, v_4, \dots, v_n$ . Then the vertex set  $V = \{v_1, v_2, v_3, v_4, \dots, v_n\}$ . Label the vertices  $v_1$  and  $v_i$  ( $i = 2, 3, \dots, n$  with  $\{1\}$ ). Then,  $S(v_1)$  is sum of the labels of the vertices of cycle that are joined to the single vertex and  $v_1$  and  $S(v_2)$  to  $S(v_n)$  is  $\{3, 4\}$ .

This induces an Inclusive lucky labeling of  $Gr_n$  and  $\eta_l(Gr_n) = 1$ .

**III. CONCLUSION**

In this paper the Inclusive lucky numbers are found for the Crown graph, Helm graph, Split graph, Book graph, Amal graph, Gear graph With the increasing importance of labeling in graph theory the study can be extended for many other types of graphs and algorithms can also be developed for the same.

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