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Special Functions (Bessel Function, Zeta Function Etc.)

MANJU SHARMA

Associate Professor, Maths, Government College, Kota, Rajasthan, India

ABSTRACT: Special functions are particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, or other applications.

The term is defined by consensus, and thus lacks a general formal definition, but the list of mathematical functions contains functions that are commonly accepted as special.

Many special functions appear as solutions of differential equations or integrals of elementary functions. Therefore, tables of integrals^[1] usually include descriptions of special functions, and tables of special functions^[2] include most important integrals; at least, the integral representation of special functions. Because symmetries of differential equations are essential to both physics and mathematics, the theory of special functions is closely related to the theory of Lie groups and Lie algebras, as well as certain topics in mathematical physics.

Symbolic computation engines usually recognize the majority of special functions.

KEYWORDS: special, functions, Bessel, Zeta, notations, etc.

I. INTRODUCTION

Most special functions are considered as a function of a complex variable. They are analytic; the singularities and cuts are described; the differential and integral representations are known and the expansion to the Taylor series or asymptotic series are available. In addition, sometimes there exist relations with other special functions; a complicated special function can be expressed in terms of simpler functions. Various representations can be used for the evaluation; the simplest way to evaluate a function is to expand it into a Taylor series. However, such representation may converge slowly or not at all. In algorithmic languages, rational approximations are typically used, although they may behave badly in the case of complex argument(s). [1,2,3,4]

History of special functions

Classical theory

While trigonometry and exponential functions were systematized and unified by the eighteenth century, the search for a complete and unified theory of special functions has continued since the nineteenth century. The high point of special function theory in 1800–1900 was the theory of elliptic functions; treatises that were essentially complete, such as that of Tannery and Molk,^[3] expounded all the basic identities of the theory using techniques from analytic function theory (based on complex analysis). The end of the century also saw a very detailed discussion of spherical harmonics. [5,6,7,8]

Changing and fixed motivations

While pure mathematicians sought a broad theory deriving as many as possible of the known special functions from a single principle, for a long time the special functions were the province of applied mathematics. Applications to the physical sciences and engineering determined the relative importance of functions. Before electronic computation, the importance of a special function was affirmed by the laborious computation of extended tables of values for ready look-up, as for the familiar logarithm tables. (Babbage's difference engine was an attempt to compute such tables.) For this purpose, the main techniques are: [9,10,11,12]

- numerical analysis, the discovery of infinite series or other analytical expressions allowing rapid calculation; and
- reduction of as many functions as possible to the given function.



More theoretical questions include: asymptotic analysis; analytic continuation and monodromy in the complex plane; and symmetry principles and other structural equations.

Twentieth century

The twentieth century saw several waves of interest in special function theory. The classic Whittaker and Watson (1902) textbook^[4] sought to unify the theory using complex analysis; the G. N. Watson tome A Treatise on the Theory of Bessel Functions pushed the techniques as far as possible for one important type, including asymptotic results.

The later Bateman Manuscript Project, under the editorship of Arthur Erdélyi, attempted to be encyclopedic, and came around the time when electronic computation was coming to the fore and tabulation ceased to be the main issue.^[13,14,15,16]

Contemporary theories

The modern theory of orthogonal polynomials is of a definite but limited scope. Hypergeometric series, observed by Felix Klein to be important in astronomy and mathematical physics,^[5] became an intricate theory, in need of later conceptual arrangement. Lie groups, and in particular their representation theory, explain what a spherical function can be in general; from 1950 onwards substantial parts of classical theory could be recast in terms of Lie groups. Further, work on algebraic combinatorics also revived interest in older parts of the theory. Conjectures of Ian G. Macdonald helped to open up large and active new fields with the typical special function flavour. Difference equations have begun to take their place besides differential equations as a source for special functions.^[17,18,19,20]

Special functions in number theory

In number theory, certain special functions have traditionally been studied, such as particular Dirichlet series and modular forms. Almost all aspects of special function theory are reflected there, as well as some new ones, such as came out of the monstrous moonshine theory.

Special functions of matrix arguments

Analogues of several special functions have been defined on the space of positive definite matrices, among them the power function which goes back to Atle Selberg,^[6] the multivariate gamma function,^[7] and types of Bessel functions.^[8]

The NIST Digital Library of Mathematical Functions has a section covering several special functions of matrix arguments.^[9]

II. DISCUSSION

Bessel functions, first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation, for an arbitrary complex number, which represents the order of the Bessel function. ^[21,22,23,24]

Bessel functions for integer are also known as cylinder functions or the cylindrical harmonics because they appear in the solution to Laplace's equation in cylindrical coordinates. Spherical Bessel functions with half-integer are obtained when solving the Helmholtz equation in spherical coordinates.

Applications of Bessel functions

The Bessel function is a generalization of the sine function. It can be interpreted as the vibration of a string with variable thickness, variable tension (or both conditions simultaneously); vibrations in a medium with variable properties; vibrations of the disc membrane, etc.

Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates. Bessel functions are therefore especially important for many problems of wave propagation and static potentials. In solving problems in cylindrical coordinate systems, one obtains Bessel functions of integer order ($\alpha = n$); in spherical problems, one obtains half-integer orders ($\alpha = n + 1/2$). For example:^[25,26,27,28]

- Electromagnetic waves in a cylindrical waveguide
- Pressure amplitudes of inviscid rotational flows



- Heat conduction in a cylindrical object
- Modes of vibration of a thin circular or annular acoustic membrane (such as a drumhead or other membranophone) or thicker plates such as sheet metal (see Kirchhoff–Love plate theory, Mindlin–Reissner plate theory)
- Diffusion problems on a lattice
- Solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates) for a free particle
- Solving for patterns of acoustical radiation
- Frequency-dependent friction in circular pipelines
- Dynamics of floating bodies[29,30,31,32]
- Angular resolution
- Diffraction from helical objects, including DNA
- Probability density function of product of two normally distributed random variables^[1]
- Analyzing of the surface waves generated by microtremors, in geophysics and seismology.

Bessel functions also appear in other problems, such as signal processing (e.g., see FM audio synthesis, Kaiser window, or Bessel filter).[33,34,35,36]

Definitions

Because this is a second-order linear differential equation, there must be two linearly independent solutions. Depending upon the circumstances, however, various formulations of these solutions are convenient. Different variations are summarized in the table below and described in the following sections.[37,38,39,40]

Type	First kind	Second kind
Bessel functions	J_α	Y_α
Modified Bessel functions	I_α	K_α
Hankel functions	$H_\alpha^{(1)}$ $_\alpha = J_\alpha + iY_\alpha$	$H_\alpha^{(2)}$ $_\alpha = J_\alpha - iY_\alpha$
Spherical Bessel functions	j_n	y_n
Spherical Hankel functions	$h_n^{(1)}$ $_n = j_n + iy_n$	$h_n^{(2)}$ $_n = j_n - iy_n$

Bessel functions of the second kind and the spherical Bessel functions of the second kind are sometimes denoted by N_n and n_n , respectively, rather than Y_n and y_n .^{[2][3]}

Bessel functions of the first kind: J_α

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation. For integer or positive α , Bessel functions of the first kind are finite at the origin ($x = 0$); while for negative non-integer α , Bessel functions of the first kind diverge as x approaches zero. It is possible to define the function by its series expansion around $x = 0$, which can be found by applying the Frobenius method to Bessel's equation.[41,42,43,44]

For non-integer α , the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent, and are therefore the two solutions of the differential equation. On the other hand, for integer order n , the following relationship is valid (the gamma function has simple poles at each of the non-positive integers):^[5]

The Bessel functions of the second kind, denoted by $Y_\alpha(x)$, occasionally denoted instead by $N_\alpha(x)$, are solutions of the Bessel differential equation that have a singularity at the origin ($x = 0$) and are multivalued. These are sometimes called Weber functions, as they were introduced by H. M. Weber (1873), and also Neumann functions after Carl Neumann.^[14]

The Hankel functions are used to express outward- and inward-propagating cylindrical-wave solutions of the cylindrical wave equation, respectively (or vice versa, depending on the sign convention for the frequency).[45,46,47,48]

III. RESULTS

The Riemann zeta function plays a pivotal role in analytic number theory, and has applications in physics, probability theory, and applied statistics.

Leonhard Euler first introduced and studied the function over the reals in the first half of the eighteenth century. Bernhard Riemann's 1859 article "On the Number of Primes Less Than a Given Magnitude" extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation,[49,50,51,52] and established a relation between its zeros and the distribution of prime numbers. This paper also contained the Riemann hypothesis, a conjecture about the distribution of complex zeros of the Riemann zeta function that many mathematicians consider the most important unsolved problem in pure mathematics.^[3]

The values of the Riemann zeta function at even positive integers were computed by Euler. The first of them, $\zeta(2)$, provides a solution to the Basel problem. In 1979 Roger Apéry proved the irrationality of $\zeta(3)$. The values at negative integer points, also found by Euler, are rational numbers and play an important role in the theory of modular forms. Many generalizations of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are known. The critical strip of the Riemann zeta function has the remarkable property of universality. This zeta function universality states that there exists some location on the critical strip that approximates any holomorphic function arbitrarily well. Since holomorphic functions are very general, this property is quite remarkable. The first proof of universality was provided by Sergei Mikhailovitch Voronin in 1975.^[22] More recent work has included effective versions of Voronin's theorem^[23] and extending it to Dirichlet L-functions[53,54,55,56]

The zeta function occurs in applied statistics (see Zipf's law and Zipf–Mandelbrot law).

Zeta function regularization is used as one possible means of regularization of divergent series and divergent integrals in quantum field theory. In one notable example, the Riemann zeta function shows up explicitly in one method of calculating the Casimir effect. The zeta function is also useful for the analysis of dynamical systems.^[46]

In mathematics, the arithmetic zeta function is a zeta function associated with a scheme of finite type over integers. The arithmetic zeta function generalizes the Riemann zeta function and Dedekind zeta function to higher dimensions. The arithmetic zeta function is one of the most-fundamental objects of number theory.[57,58,59,60]

The arithmetic zeta function of a regular connected equidimensional arithmetic scheme of Kronecker dimension n can be factorized into the product of appropriately defined L-factors and an auxiliary factor. Hence, results on L-functions imply corresponding results for the arithmetic zeta functions. However, there is still very little amount of proven results about the L-factors of arithmetic schemes in characteristic zero and dimensions 2 and higher. Ivan Fesenko initiated^[5] a theory which studies the arithmetic zeta functions directly, without working with their L-factors. It is a higher-dimensional generalisation of Tate's thesis, i.e. it uses higher adèle groups, higher zeta integral and objects which come from higher class field theory.[61,62,63,64] In this theory, the meromorphic continuation and functional equation of proper regular models of elliptic curves over global fields is related to mean-periodicity property of a boundary function.^[6] In his joint work with M. Suzuki and G. Ricotta a new correspondence in number theory is proposed, between the arithmetic zeta functions and mean-periodic functions in the space of smooth functions on the real line of not more than exponential growth.^[7] This correspondence is related to the Langlands correspondence. Two other applications of Fesenko's theory are to the poles of the zeta function of proper models of elliptic curves over global fields and to the special value at the central point.^[8]

IV. CONCLUSION

ZetaGrid was at one time the largest distributed computing project, designed to explore the non-trivial roots of the Riemann zeta function, checking over one billion roots a day.[73,74]

Roots of the zeta function are of particular interest in mathematics; a single root out of alignment would disprove the Riemann hypothesis, with far-reaching consequences for all of mathematics. As of June 2022, no counterexample to the Riemann hypothesis has been found.[65,66,67,68]

The project ended in November 2005 due to instability of the hosting provider.^[1] The first more than 10^{13} zeroes were checked.^[2] The project administrator stated that after the results were analyzed, they would be posted on the American Mathematical Society website.^[3] The official status remains unclear, however, as it was never published nor independently verified. This is likely because there was no evidence that each zero was actually computed, as there was no process implemented to check each one as it was calculated[69,70,71,72]

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