



e-ISSN: 2319-8753 | p-ISSN: 2347-6710

# IJIRSET

International Journal of Innovative Research in  
**SCIENCE | ENGINEERING | TECHNOLOGY**



# INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH

IN SCIENCE | ENGINEERING | TECHNOLOGY

Volume 13, Issue 7, July 2024

**ISSN** INTERNATIONAL  
STANDARD  
SERIAL  
NUMBER  
INDIA

Impact Factor: 8.423

 9940 572 462

 6381 907 438

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# Unique Common Fixed Point Theorem for Four Maps Rational Type Contraction in Symmetric Spaces

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**ABSTRACT:** In this paper we obtain a unique common fixed point theorem for four mappings satisfying a rational type contractive condition in symmetric spaces. Also we give an example to illustrate our theorem.

**KEYWORDS:** symmetric space, weakly compatible.

## I. INTRODUCTION AND PRELIMINARIES

The notions of symmetric spaces and semi-metric spaces introduced and studied by Wilson [2]. Later several authors, for instance, proved some fixed, common fixed point theorems in metric space. (see, [[3],[4]]). In 2015, Almeida, Roldan-Lopez-de-Hiero and sadarangani [5] proved that whenever  $f$  is a rational type contraction mapping from a complete metric space into itself, then  $f$  has a unique fixed point.

In this paper we prove unique common fixed point theorem for four mappings satisfying a rational type contractive condition in symmetric spaces. Also we give an example to illustrate our theorem.

First we state the following known definitions, lemmas and propositions.

**Definition 1.1** [6]. Suppose that  $X$  be a non-empty set and  $S: X \times X \rightarrow [0, \infty)$  be a distance function such that:

- $S(x, y) = 0 \Leftrightarrow x = y$ ,
- $S(x, y) = S(y, x)$ , for
- all  $x, y \in X$ .

We mean by a pair  $(X, S)$  with a symmetric space.

**Definition 1.2**[6]. Let  $(X, S)$  be a symmetric space.

a) A sequence  $\{x_n\}$  in  $X$  is  $S$ -Cauchy sequence if  $\lim_{n \rightarrow \infty} S(x_n, x_{n+1}) = 0, r \in \mathbb{N}$  (the set of all natural numbers).

b)  $(X, S)$  is  $S$ -complete iff or every  $S$ -Cauchy sequence  $\{x_n\}$ , there exists  $x$  in  $X$  with  $\lim_{n \rightarrow \infty} S(x_n, x) = 0$ .

c)  $f: X \rightarrow X$  is  $S$ -continuous if  $\lim_{n \rightarrow \infty} S(x_n, x) = 0$  implies

$$\lim_{n \rightarrow \infty} S(fx_n, fx) = 0.$$

We need the following properties in a symmetric space  $(X, S)$ .  $(W_3)$ [2] Given  $\{x_n\}$ ,  $y$  and  $x$  in  $X$ ,  $\lim_{n \rightarrow \infty} S(x_n, y) = 0$  and  $\lim_{n \rightarrow \infty} S(x_n, x) = 0$  implies that  $x = y$ .

$(W_4)$  [2] Given  $\{x_n\}$ ,  $\{y_n\}$ , and  $x$  in  $X$ ,  $\lim_{n \rightarrow \infty} S(x_n, y) = 0$  and  $\lim_{n \rightarrow \infty} S(x_n, y_n) = 0$  imply that

$$\lim_{n \rightarrow \infty} S(y_n, x) = 0.$$

$(1C)$ [7] A function  $S$  is 1-continuous if

$$\lim_{n \rightarrow \infty} S(x_n, x) = 0 \Rightarrow \lim_{n \rightarrow \infty} S(x_n, y) = 0 = S(x, y).$$

**Remark 1.3**[2].  $(W_4) \Rightarrow (W_3)$ .

**Proposition 1.4** [1]. Let  $(X, S)$  be symmetric space. Then the following conditions are equivalent.

1.  $(X, S)$  satisfies property (JMS).
2. There exists  $\delta, \eta > 0$  such that for any  $x, y, z \in X$ ,  
 $S(x, z) + S(z, y) < \delta$  implies that  $d(x, y) < \eta$ .
3. There exists  $r > 0$  such that

$$\sup \{ \text{daim}(B(x, r)) : x \in X \} < \infty$$

## II. MAIN RESULTS

Now we give our main results.

**Theorem 2.1** Suppose that  $(X, S)$  be a  $S$ -complete symmetric space satisfy (W4) and (1C). Let  $F, G, f$  and  $g$  be a self mappings on  $X$  and the following condition holds.

(2.1.1)

$$S(Fx, Gy) \leq \phi(M(x, y)) + c \min \left\{ \begin{matrix} S(fx, Fx), S(gy, Gy), \\ S(fx, Gy), S(gy, Fx) \end{matrix} \right\} \forall x, y \in X, c \geq 0$$

where

$$M(x, y) = \max \left\{ S(fx, gy), \frac{S(fx, Fx)[1 + S(gy, Gy)]}{1 + S(fx, gy)}, \frac{S(gy, Gy)[1 + S(fx, Fx)]}{1 + S(fx, gy)} \right\}$$

...(1)

(2.1.2)  $FX \subseteq gX, GX \subseteq fX$ ,

(2.1.3)  $(F, f)$  and  $(G, g)$  are weakly compatible,

(2.1.4)  $fX$  or  $gX$  is a complete subspace of  $X$ , and  $\phi: [0, \infty) \rightarrow [0, \infty)$  be a continuous, nondecreasing function and  $\lim_{n \rightarrow \infty} \phi^n(t) = 0, \forall t > 0$

Then  $F, G, f$  and  $g$  have unique common fixed point.

**Proof.** Let  $x_0 \in X$  be an arbitrary point. Using condition (2.1.2) We define a sequence  $\{y_n\}$  in  $X$  such that  $y_{2n+1} = gx_{2n+1} = Fx_{2n}$  and

$$y_{2n+2} = fx_{2n+2} = Gx_{2n+1}, n = 0, 1, 2, \dots$$

Then

$$S(y_{2n+1}, y_{2n+2}) = S(Fx_{2n}, Gx_{2n+1})$$

$$\leq \phi(M(x_{2n}, x_{2n+1})) + c \min \left\{ \begin{matrix} S(fx_{2n}, Fx_{2n}), S(gx_{2n+1}, Gx_{2n+1}), \\ S(fx_{2n}, Gx_{2n+1}), S(gx_{2n+1}, Fx_{2n}) \end{matrix} \right\}$$

$$= \phi(M(x_{2n}, x_{2n+1})) + c \min \left\{ \begin{matrix} S(y_{2n}, y_{2n+1}), S(y_{2n+1}, y_{2n+2}), \\ S(y_{2n}, y_{2n+2}), S(y_{2n+1}, y_{2n+1}) \end{matrix} \right\}$$

$$= \phi(M(x_{2n}, x_{2n+1})) + c \min \left\{ \begin{matrix} S(y_{2n}, y_{2n+1}), S(y_{2n+1}, y_{2n+2}), \\ S(y_{2n}, y_{2n+2}), 0 \end{matrix} \right\}$$

$$= \phi(M(x_{2n}, x_{2n+1})) \dots \dots \dots (2)$$

Where

$$M(x_{2n}, x_{2n+1}) = \max \left\{ \begin{matrix} S(fx_{2n}, gx_{2n+1}), \frac{S(fx_{2n}, Fx_{2n})[1 + S(gx_{2n+1}, Gx_{2n+1})]}{1 + S(fx_{2n}, gx_{2n+1})}, \\ \frac{S(gx_{2n+1}, Gx_{2n+1})[1 + S(fx_{2n}, Fx_{2n})]}{1 + S(fx_{2n}, gx_{2n+1})} \end{matrix} \right\}$$

$$= \max \left\{ \begin{matrix} S(y_{2n}, y_{2n+1}), \frac{S(y_{2n}, y_{2n+1})[1 + S(y_{2n+1}, y_{2n+2})]}{1 + S(y_{2n}, y_{2n+1})}, \\ S(y_{2n+1}, y_{2n+2}) \end{matrix} \right\}$$

If  $M(x_{2n}, x_{2n+1}) = S(y_{2n}, y_{2n+1})$ , form (2), we have

$$S(y_{2n+1}, y_{2n+2}) \leq \phi(M(x_{2n}, x_{2n+1}))$$

$$= \phi(S(y_{2n}, y_{2n+1}))$$

$$S(y_{2n+1}, y_{2n+2}) < S(y_{2n}, y_{2n+1}) \dots \dots \dots (3)$$

$$\text{If } M(x_{2n}, x_{2n+1}) = \frac{S(y_{2n}, y_{2n+1})[1 + S(y_{2n+1}, y_{2n+2})]}{1 + S(y_{2n}, y_{2n+1})}$$

From (2) we obtain

$$S(y_{2n+1}, y_{2n+2}) \leq \phi \left( \frac{S(y_{2n}, y_{2n+1})[1 + S(y_{2n+1}, y_{2n+2})]}{1 + S(y_{2n}, y_{2n+1})} \right)$$

$$< \frac{S(y_{2n}, y_{2n+1})[1 + S(y_{2n+1}, y_{2n+2})]}{1 + S(y_{2n}, y_{2n+1})}$$

Hence  $S(y_{2n+1}, y_{2n+2}) < S(y_{2n}, y_{2n+1})$

That is (3) holds.

If  $M(x_{2n}, x_{2n+1}) = S(y_{2n+1}, y_{2n+2})$  from (2), we get

$$S(y_{2n+1}, y_{2n+2}) < S(y_{2n+1}, y_{2n+2})$$

Which is impossible. In any case, we proved that(3) holds.

Since  $S(y_{2n+1}, y_{2n+2})$  is decreasing. Hence it converges to a non negative number,

$c \geq 0$ . If  $c > 0$  then letting  $n \rightarrow \infty$  in(2),

$$\text{we deduce } c \leq \phi \left( \max \left\{ c, \frac{c(1+c)}{1+c}, c \right\} \right) = \phi(c) < c.$$

Which implies that  $c=0$ , that is

$$\lim_{n \rightarrow \infty} S(y_{2n+1}, y_{2n+2}) = 0$$

Then there exists  $k \in \mathbb{N}$  such that  $S\{y_{2k}, y_{2k+1}\} < \min \left\{ \frac{\delta}{2}, \eta \right\}$ .

Now we will prove that, for all  $m \in \mathbb{N}$

$$S\{y_{2k}, y_{2k+m}\} < \eta$$

By the definition of  $k$ , we get that (1) is valid for  $m=1$ . Thus

$$S\{y_{2k}, y_{2k+1}\} < \frac{\delta}{2}$$

Now we assume that (1) is satisfied for some  $m=n \in \mathbb{N}$ . Thus

$$S\{y_{2k+1}, y_{2k+n+1}\} < \phi(S\{y_{2k}, y_{2k+n}\}) < \phi(\eta) < \frac{\delta}{2}$$

It follows that

$$S\{y_{2k}, y_{2k+1}\} + S\{y_{2k+1}, y_{2k+n+1}\} < \delta$$

Which by Proposition 1.1, implies that

$$S\{y_{2k}, y_{2k+n+1}\} < \eta$$

So by induction, we have that (1) is satisfied for any  $n \geq 1$ . That is

$$S\{y_{2k}, y_{2k+m}\} \leq \phi^n(\eta)$$

Therefore  $\{y_{2n}\}$  is a S-cauchy sequence. Similarly we can also prove that  $\{y_{2n+1}\}$  is S-cauchy sequence. Hence

$\{y_n\}$  S-cauchy sequence. Since  $(X, S)$  is a

S- complete, there exists  $z \in X$  such that  $y_n \rightarrow z$  as  $n \rightarrow \infty$ .

Thus

$$\lim_{n \rightarrow \infty} Fx_{2n} = \lim_{n \rightarrow \infty} gx_{2n+1} = \lim_{n \rightarrow \infty} Gx_{2n+1} = \lim_{n \rightarrow \infty} Fx_{2n+2} = z$$

Now if  $fX$  is complete subspace of  $X$ , there exists  $u \in X$  such that  $fu=z$ .

Now from condition (2.1.1)

$$S(Fu, z) = S(Fu, Gx_{2n+1})$$

$$\leq \phi(M(u, x_{2n+1})) + c \min \left\{ S(fu, Fu), S(gx_{2n+1}, Gx_{2n+1}), S(fu, Gx_{2n+1}), S(gx_{2n+1}, Fu) \right\}$$

$$= \phi(M(u, x_{2n+1})) + c \min \left\{ S(z, Fu), S(y_{2n+1}, y_{2n+2}), S(z, y_{2n+2}), S(y_{2n+1}, Fu) \right\}$$

$$= \phi(M(u, x_{2n+1})) + c \min \{ S(z, Fu), 0, 0, S(z, Fu) \}$$

$$= \phi(M(u, x_{2n+1})). \dots \dots \dots (4)$$

Where

$$M(u, x_{2n+1})$$

$$= \max \left\{ S(fu, gx_{2n+1}), \frac{S(fu, Fu)[1 + S(gx_{2n+1}, Gx_{2n+1})]}{1 + S(fu, gx_{2n+1})}, \frac{S(gx_{2n+1}, Gx_{2n+1})[1 + S(fu, Fu)]}{1 + S(fu, gx_{2n+1})} \right\}$$

$$= \max \left\{ S(z, z), \frac{S(z, Fu)[1 + S(z, z)]}{1 + S(z, z)}, \frac{S(z, z)[1 + S(z, Fu)]}{1 + S(z, z)} \right\}$$

$$= \max\{0, S(z, Fu), 0\}$$

$$= S(z, Fu)$$

From(4)

$$S(Fu, z) \leq \phi(M(u, x_{2n+1}))$$

$$= \phi(S(z, Fu))$$

$$= S(z, Fu)$$

Which is contradiction. Hence  $S(Fu, z) = 0$  That is  $Fu = z$ .  $z$  is a fixed point of  $F$ . Thus  $fu = Fu = z$ . Since  $Fx \subset gx$ , there exists  $v \in X$  such that  $Fu = gv$ . Thus  $fu = Fu = gv = z$ .

Again from condition(2.1.1)

$$S(z, Gv) = S(Fu, Gv)$$

$$\leq \phi(M(u, v)) + c \min\{S(fu, Fu), S(gv, Gv), S(fu, Gv), S(gv, Fu)\}$$

$$= \phi(M(u, v)) + c \min\{S(z, z), S(z, Gv), S(z, Gv), S(z, z)\}$$

$$= \phi(M(u, v)) + c \min\{0, S(z, Gv), 0\}$$

$$= \phi(M(u, v)) \dots \dots \dots (5)$$

Where

$$M(u, v) = \max \left\{ S(fu, gv), \frac{S(fu, Fu)[1 + S(gv, Gv)]}{1 + S(fu, gv)}, \frac{S(gv, Gv)[1 + S(fu, Fu)]}{1 + S(fu, gv)} \right\}$$

$$= \max \left\{ S(z, z), \frac{S(z, z)[1 + S(z, Gv)]}{1 + S(z, z)}, \frac{S(z, Gv)[1 + S(z, z)]}{1 + S(z, z)} \right\}$$

$$= \max\{0, 0, S(z, Gv)\}$$

$$= S(z, Gv)$$

From (5),

$$S(z, Gv) \leq \phi(M(u, v))$$

$$= \phi(S(z, Gv))$$

$$= S(z, Gv)$$

Which is contradiction. Hence

$S(z, Gv) = 0$  That is  $Gv = z$ .  $z$  is a fixed point of  $G$ . Thus  $fu = Fu = z = gv = Gv$ .

Since  $f$  and  $F$  are weakly compatible  $fz = fFu = Ffu = Fz$   $S(Fz, z) = S(Fz, Gv)$

$$\leq \phi(M(z, v)) + c \min\{S(fz, Fz), S(gv, Gv), S(fz, Gv), S(gv, Fz)\}$$

$$= \phi(M(z, v)) + c \min\{S(fz, fz), S(z, z), S(Fz, z), S(z, Fz)\}$$

$$= \phi(M(z, v)) + c \min\{0, 0, S(Fz, z), S(z, Fz)\}$$

$$= \phi(M(z, v)) \dots \dots \dots (6)$$

Where

$$M(z, v) = \max \left\{ S(fz, gv), \frac{S(fz, Fz)[1 + S(gv, Gv)]}{1 + S(fz, gv)}, \frac{S(gv, Gv)[1 + S(fz, Fz)]}{1 + S(fz, gv)} \right\}$$

$$= \max \left\{ S(Fz, z), \frac{S(Fz, fz)[1 + S(z, z)]}{1 + S(z, z)}, \frac{S(z, z)[1 + S(z, z)]}{1 + S(z, z)} \right\}$$

$$= \max \{ S(Fz, z), 0, 0 \}$$

$$= S(Fz, z)$$

from(6), we obtain

$$S(Fz, z) \leq \phi(M(z, v))$$

$$= \phi(S(Fz, z))$$

$$= S(Fz, z)$$

Which is contradiction. Hence  $S(Fz, z)=0$ .

That is  $Fz=z=fz$ .

Thus  $g$  and  $T$  are weakly compatible  $gz=gGv=Ggv=Gz$ .

Also

$$S(z, Gz) = S(Fz, Gz)$$

$$\leq \phi(M(z, z)) + c \min \left\{ \begin{matrix} S(fz, Fz), S(gz, Gz), \\ S(fz, Gz), S(gz, Fz) \end{matrix} \right\}$$

$$= \phi(M(z, z)) + c \min \left\{ \begin{matrix} S(z, z), S(Gz, Gz), \\ S(z, Gz), S(Gz, z) \end{matrix} \right\}$$

$$= \phi(M(z, z)) + c \min \{ 0, 0, S(z, Gz) \}$$

$$= \phi(M(z, z)) \dots\dots\dots(7)$$

where

$$M(z, z) = \max \left\{ S(fz, gz), \frac{S(fz, Fz)[1 + S(gz, Gz)]}{1 + S(fz, gz)}, \frac{S(gz, Gz)[1 + S(fz, Fz)]}{1 + S(fz, gz)} \right\}$$

$$= \max \{ S(z, Gz), 0, 0 \}$$

$$= S(z, Gz)$$

Form 7 we obtain

$$S(z, Gz) \leq \phi(M(z, z))$$

$$= \phi(S(z, Gz))$$

$$= S(z, Gz)$$

Which is contradiction. Hence  $S(z, Gz)=0$ . That is  $Gz= z$ .

Thus  $gz= Gz= z$ . Thus  $fz = Fz = z= gz= Gz$ . Thus  $z$  is a common fixed point of  $F, G, f$  and  $g$ .

For uniqueness, let  $z^* \in X$  such that  $fz^*=Fz^*=z^*=gz^*=Gz^*$ .

Now from (i)

Also

$$S(z, z^*) = S(Fz, Gz^*)$$

$$\leq \phi(M(z, z^*)) + c \min \left\{ \begin{matrix} S(fz, Fz), S(gz^*, Gz^*), \\ S(fz, Gz^*), S(gz^*, Fz) \end{matrix} \right\}$$

$$= \phi(M(z, z^*)) + c \min \{ 0, 0, S(z, Gz^*), S(z, Gz^*) \}$$

$$= \phi(M(z, z^*)) \dots\dots\dots(8)$$

Where

$$M(z, z^*) = \max \left\{ S(fz, gz^*), \frac{S(fz, Fz)[1 + S(gz^*, Gz^*)]}{1 + S(fz, gz^*)}, \frac{S(gz^*, Gz^*)[1 + S(fz, Fz)]}{1 + S(fz, gz^*)} \right\}$$

$$= \max \{ S(z, z^*), 0, 0 \}$$

$$= S(z, z^*)$$

From(8), we obtain

$$S(z, z^*) \leq \phi(M(z, z^*))$$

$$= \phi(S(z, z^*))$$

$$= S(z, z^*)$$

Which is contradiction. Hence  $S(z, z^*)=0$ .

That is  $z = z^*$ .

Thus  $z$  is a unique common fixed point of  $F, G, f$  and  $g$ .

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