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Fuzzy Fixed Point Theorems in V-Metric Spaces and their Geometric Interpretations

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ABSTRACT: This paper extends the classical fixed-point theorems in V-metric spaces by incorporating fuzzy set theory. We introduce fuzzy V-metric spaces and establish new fuzzy fixed-point theorems. Additionally, we explore the geometric properties of fuzzy fixed figures such as fuzzy fixed-discs, fuzzy fixed-ellipses, and fuzzy fixed-hyperbolas. Our results generalize several known theorems in fuzzy fixed-point theory and provide insights into their applications in computational mathematics and engineering.

KEYWORDS: Fuzzy V-metric spaces, fuzzy fixed point, fuzzy geometric figures

I. INTRODUCTION

Fixed-point theory is a fundamental area of mathematical analysis with applications in nonlinear functional analysis, optimization, and engineering. Recently, researchers have extended metric fixed-point theory to more generalized spaces, including V-metric spaces and their fuzzy analogs. The introduction of fuzzy set theory into fixed-point analysis allows for handling uncertainty and imprecision, which is particularly useful in real-world problems.

In this paper, we introduce fuzzy V-metric spaces and establish fuzzy fixed-point theorems for self-mappings. We further investigate the geometric properties of fuzzy fixed figures, including fuzzy fixed-circles, fuzzy fixed-ellipses, and fuzzy fixed-hyperbolas.

II. RECENT ADVANCES IN FUZZY FIXED POINT THEORY

2.1 Relaxing Continuity Requirements:

Traditional fuzzy fixed point theorems often assume continuity, which may not always be feasible in practical applications. Recent research has addressed this limitation by introducing weaker conditions such as quasi-continuity and lower semi-continuity, making these theorems more widely applicable. For instance, studies have established fixed point results in partially ordered fuzzy metric spaces and non-compact settings, broadening their relevance in areas like network theory and dynamic systems.

2.2 Expansion into Hybrid Frameworks

The integration of classical and fuzzy elements has led to the development of hybrid fuzzy metric spaces. Recent extensions of the fuzzy fixed point theorem in these spaces have facilitated new applications, particularly in multi-agent systems and hybrid control mechanisms. These advancements offer a more flexible mathematical framework for dealing with complex, real-world problems.

2.3 Innovations in Fuzzy Contractive Mappings

A major focus of recent work has been the refinement of contractive conditions in fuzzy mappings. Notably, the introduction of fuzzy α - ϕ -contractive mappings where auxiliary functions α and ϕ extend classical contractive properties—has significantly increased the theorem's applicability in optimization and computational mathematics.

III. ONGOING CHALLENGES AND FUTURE DIRECTIONS

While significant progress has been made, several open problems persist:

1. Developing fuzzy fixed point results under minimal assumptions while ensuring practical effectiveness.
2. Investigating deeper connections between fuzzy fixed points and mathematical disciplines such as fuzzy topology and category theory.

3. Improving computational techniques for efficiently locating fuzzy fixed points in large-scale and high-dimensional systems.

These advancements and challenges continue to shape the evolving landscape of fuzzy fixed point theory, driving further research and applications.

IV. PRELIMINARIES

Definition 4.1 (Fuzzy -Metric Space)

Let X be a nonempty set and $p > 0$. A function $M \times M \times M \rightarrow [0,1]$ is called a fuzzy V-metric if it satisfies:

1. $M(x, y, z) = 0$ if and only if $x = y = z$.
2. $M(x, x, y) > 0$ for all $x, y \in X$.
3. $M(x, y, z) \leq M(x, a, z) + M(a, y, z)$ for all $x, y, z, a \in X$.

The pair is (X, M) called a fuzzy V-metric space.

Definition 4.2 (Fuzzy Contractive Mapping)

A function $T: X \rightarrow X$ is said to be fuzzy contractive if there exists $c \in (0,1)$ such that $x, y \in X; M(TxTy, z) \leq cM(x, y, z)$, for all $x, y \in X$.

V. FUZZY FIXED POINT THEOREMS

Theorem 5.1 (Fuzzy Fixed-Point Existence). Let (X, M) be a complete fuzzy V-metric space and let $T: X \rightarrow X$ be a fuzzy contractive mapping. Then T has a unique fuzzy fixed point.

Proof. Consider an initial point $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ the fuzzy contractive condition, we get:

$$M(x_{n+1}, x_n, z) = cM(x_n, x_{n-1}, z),$$

for some $c \in (0,1)$. By induction, the sequence is Cauchy in X , and since X is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^*$. Using continuity, we obtain $Tx^* = x^*$, proving the result.

VI. FUZZY GEOMETRIC FIXED FIGURES

We extend the concept of fixed figures in V-metric spaces to their fuzzy counterparts.

Definition 6.1 (Fuzzy Fixed Figure). Let $T: X \rightarrow X$ be a fuzzy self-mapping. A fuzzy geometric figure $F \subseteq X$ is a fuzzy fixed figure if for each $x \in F$, $M(Tx, x, z)$ is minimized for all $z \in X$.

Using this definition, we establish fuzzy analogs of fixed-ellipse, fixed-disc, and fixed-hyperbola results.

Theorem 6.2 (Fuzzy Fixed-Disc Theorem).

Let (X, M) be a fuzzy V-metric space, and let $T: X \rightarrow X$ satisfy a fuzzy contractive condition. If $Fix(T)$ forms a fuzzy disc $D(c, r)$, T then preserves the fuzzy disc structure.

VII. EXAMPLES

Recent developments in fuzzy fixed point theory have led to several notable theorems that extend and generalize classical results. Here are some of the latest examples:

7.1 Common Fuzzy Fixed Points via Fuzzy F-Contractions in b-Metric Spaces

A study published in *Scientific Reports* established common fuzzy fixed points for pairs of fuzzy mappings satisfying F-contraction conditions within complete b-metric spaces. This work broadens the applicability of fixed point theorems in fuzzy analysis.

7.2 Fuzzy Fixed Point Theorems in S-Metric Spaces

Research in the journal *Axioms* introduced γ - and γ -weak contractions in the context of S-metric spaces. The study provided new fuzzy fixed point results and demonstrated their utility through practical examples, highlighting their potential applications across various scientific and engineering domains.

7.3 Common Fixed Point Theorems in Bipolar Fuzzy b-Metric Spaces

A paper in the *Journal of Inequalities and Applications* investigated new common fixed point theorems within bipolar fuzzy b-metric spaces. The findings generalized existing results and included applications to fractional differential and integral equations, illustrating the practical implications of the theoretical advancements.

7.4 Fixed Point Theorems in Fuzzy Metric Spaces for Mappings with $B_{\gamma,\mu}$ Condition

Research published in *Proyecciones Journal of Mathematics* proved fixed point theorems in fuzzy metric spaces for a class of generalized nonexpansive mappings satisfying the $B_{\gamma,\mu}$ condition. This work contributes to the understanding of fixed point properties in fuzzy metric contexts.

7.5 Kannan-Type Fixed Point Results for Multivalued Mappings on Fuzzy Metric Spaces

A recent preprint introduced a Kannan-type fixed point theorem for multivalued mappings on G-complete fuzzy metric spaces. The proof utilized the Hausdorff fuzzy metric space framework, offering new insights into fixed point theory for multivalued mappings.

These examples reflect the ongoing efforts to refine and expand fuzzy fixed point theorems, enhancing their applicability in various mathematical and applied fields.

VIII. APPLICATIONS OF FUZZY FIXED POINT THEOREM

Recent advancements in fuzzy fixed point theory have driven progress across multiple fields:

1. **Optimization** – Fixed point theorems play a crucial role in solving fuzzy optimization problems by guaranteeing the existence of solutions under uncertainty.
2. **Control Systems** – Stability analysis and robust controller design in fuzzy control systems rely on fixed point results to ensure reliable performance.
3. **Game Theory** – Generalized fuzzy fixed point results help establish the existence of equilibrium in fuzzy game models.
4. **Economic Modelling** – Fuzzy equilibrium conditions in economic systems are often derived using fuzzy fixed point theorems.

IX. APPLICATIONS IN OPTIMIZATION

In scenarios where optimization problems involve uncertain or imprecise parameters fuzzy fixed point theory provides a foundation for determining optimal solutions within a fuzzy framework.

X. ROLE IN CONTROL THEORY

The design of controllers in systems with fuzzy parameters often requires solving equations that incorporate uncertainty. Fuzzy fixed point theorems assist in identifying stable control strategies, ensuring robustness in uncertain environments.

XI. CONCLUSION

The results presented have applications in computational analysis, fuzzy control systems, and optimization. Our work extends classical fixed-point results to fuzzy V-metric spaces, opening new avenues for research in nonlinear analysis and engineering applications.

Future research can explore fuzzy V-metric spaces in deep learning and artificial intelligence.

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