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# Application of Fuzzy Logic to Rectangular Metric Spaces using Fixed Point Theorems

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**ABSTRACT:** This research explores the application of fuzzy logic to rectangular metric spaces using fixed-point theorems. The concept of a rectangular metric space extends traditional metric spaces by relaxing the triangle inequality condition. We investigate the existence and uniqueness of fixed points for fuzzy mappings within these spaces. The study highlights the role of fuzzy contractive conditions and their implications for stability and convergence in mathematical analysis.

## I. INTRODUCTION

Fixed-point theory is a crucial area of mathematical analysis with applications in various disciplines such as optimization, differential equations, and control theory. The classical Banach contraction principle has been widely extended to generalized metric spaces, including rectangular metric spaces. Simultaneously, fuzzy set theory has provided a powerful framework for handling uncertainty and imprecision in mathematical modeling.

The concept of a rectangular metric space provides a generalization of classical metric spaces by introducing a weaker form of the triangle inequality, which facilitates broader applications in nonlinear analysis and computational models. Many real-world phenomena involve uncertainty, and fuzzy mappings offer a natural tool to model such scenarios. Thus, investigating fixed points of fuzzy mappings in rectangular metric spaces allows us to develop a more comprehensive theoretical foundation with practical applications in decision-making, image processing, and artificial intelligence.

This paper aims to bridge the gap between fuzzy logic and rectangular metric spaces by establishing novel fixed-point results under fuzzy contractive conditions. We also present several examples to illustrate our findings and discuss potential applications.

## II. PRELIMINARIES

**2.1 Rectangular Metric Spaces** A rectangular metric space  $(X, d)$  is a generalization of a metric space where the distance function  $d: X \times X \rightarrow \mathbb{R}^+$  satisfies the following properties:

1.  $d(x, y) = 0$  if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$  for all  $x, y \in X$  (symmetry).
3. There exists a constant  $\alpha > 0$  such that for all distinct  $x, y, z, w \in X$ :  $d(x, y) \leq \alpha[d(x, z) + d(z, w) + d(w, y)]$  This condition serves as a generalization of the classical triangle inequality and allows more flexibility in distance measurement.

Rectangular metric spaces are particularly useful in studying fixed points of mappings that do not satisfy standard contraction conditions but still exhibit convergence behavior.

**2.2 Fuzzy Mappings** Fuzzy mappings extend classical deterministic functions by associating each element  $x \in X$  with a fuzzy set  $F(x)$  rather than a single value. This approach captures uncertainty and partial membership, which are essential in fields such as artificial intelligence, decision theory, and optimization.

A fuzzy mapping  $F: X \rightarrow X$  is characterized by a membership function  $\mu: X \times X \rightarrow [0, 1]$  that determines the degree of association between elements. A common contractive condition for fuzzy mappings in a rectangular metric space is given by:  $d(F(x), F(y)) \leq \lambda d(x, y)$  for some  $\lambda \in [0, 1)$ . This ensures that the mapping brings elements closer together, ultimately leading to a fixed point.

**2.3 Notions of Completeness** A rectangular metric space  $(X, d)$  is said to be complete if every Cauchy sequence converges to a point in  $X$ . That is, if for every sequence  $\{x_n\}$  such that:  $\lim_{m,n \rightarrow \infty} d(x_m, x_n) = 0$ , there exists  $x^* \in X$  such that  $\lim_{n \rightarrow \infty} x_n = x^*$ .

The completeness of  $X$  is crucial in proving fixed-point theorems, as it guarantees that iterative sequences generated by fuzzy mappings do not escape the space.

### III. MAIN RESULTS

#### 3.1 Lemmas and Supporting Results:

**Lemma 1:** Let  $(X, d)$  be a rectangular metric space, and let  $\{x_n\}$  be a sequence in  $X$  such that  $d(x_n, x_{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\{x_n\}$  is a Cauchy sequence.

**Proof:** Given  $d(x_n, x_{n+1}) \rightarrow 0$ , for any  $\epsilon > 0$ , there exists  $N$  such that for all  $m, n > N$ , we have  $d(x_m, x_{m+1}) < \epsilon/(3\alpha)$ . Using the rectangular inequality iteratively, we obtain  $d(x_m, x_n) \leq \alpha[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \dots + d(x_{n-1}, x_n)]$  which shows  $\{x_n\}$  is Cauchy.

**Lemma 2:** If  $(X, d)$  is a complete rectangular metric space, then every decreasing sequence of closed subsets with vanishing diameter has a non-empty intersection.

**Proof:** Consider a sequence  $\{A_n\}$  of closed subsets with  $\text{diam}(A_n) \rightarrow 0$ . Since  $X$  is complete, a nested sequence of closed sets with decreasing diameter has a unique limit point, ensuring  $\bigcap A_n \neq \emptyset$ .

**Lemma 3:** Let  $F: X \rightarrow X$  be a fuzzy contractive mapping in a rectangular metric space. Then, for any sequence  $\{x_n\}$  defined by  $x_{n+1} = F(x_n)$ , the sequence is Cauchy.

**Proof:** Since  $F$  is contractive, we have:  $d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1})$ . Iterating this relation, we get:  $d(x_m, x_n) \leq \alpha \sum_{k=m}^{n-1} \lambda^k d(x_0, x_1)$ . Since  $\lambda \in [0, 1)$ , the sum converges, proving  $\{x_n\}$  is Cauchy.

**Lemma 4:** Suppose  $F$  is a fuzzy contractive mapping in a complete rectangular metric space. Then  $F$  has at most one fixed point.

**Proof:** Assume  $x^*$  and  $y^*$  are fixed points, i.e.,  $F(x^*) = x^*$  and  $F(y^*) = y^*$ . Then:  $d(x^*, y^*) = d(F(x^*), F(y^*)) \leq \lambda d(x^*, y^*)$ . Since  $\lambda \in [0, 1)$ , this implies  $d(x^*, y^*) = 0$ , proving uniqueness.

#### 3.2 Existence of Fixed Points

We establish the following theorem:

**Theorem 1:** Let  $(X, d)$  be a complete rectangular metric space and  $F: X \rightarrow X$  a fuzzy contractive mapping with a contractivity constant  $\lambda \in [0, 1)$ . Then  $F$  has a unique fixed point, i.e., there exists  $x^* \in X$  such that  $F(x^*) = x^*$ .

**Proof:** Consider an arbitrary point  $x_0 \in X$  and define a sequence  $x_{n+1} = F(x_n)$ . Using the contractive condition,  $d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1})) \leq \lambda d(x_n, x_{n-1})$ . By induction, we obtain:  $d(x_n, x_{n-1}) \leq \lambda^n d(x_1, x_0)$ . Since  $\lambda < 1$ , the sequence  $\{x_n\}$  is Cauchy and converges to some  $x^*$ . Taking limits on both sides, we obtain  $F(x^*) = x^*$ , proving the existence of a fixed point.

**Theorem 2:** Let  $(X, d)$  be a rectangular metric space and  $F: X \rightarrow X$  be a fuzzy mapping satisfying the inequality:  $d(F(x), F(y)) \leq \beta d(x, y) + \gamma d(F(x), x)$  for some constants  $0 \leq \beta, \gamma < 1$ . Then  $F$  has a unique fixed point.

**Proof:** Consider a sequence  $x_{n+1} = F(x_n)$ . Applying the given contractive condition recursively,  $d(x_{n+1}, x_n) \leq \beta d(x_n, x_{n-1}) + \gamma d(F(x_n), x_n)$ . Since both terms on the right-hand side are shrinking due to  $\beta, \gamma < 1$ , the sequence  $\{x_n\}$  is Cauchy and converges to some  $x^*$ , satisfying  $F(x^*) = x^*$ .

**Theorem 3:** Let  $(X, d)$  be a rectangular metric space with a fuzzy mapping  $F: X \rightarrow X$  satisfying the condition:  $d(F(x), F(y)) \leq \alpha d(x, y) + (1 - \alpha)d(F(x), F(y))$  where  $0 < \alpha < 1$ . Then  $F$  has a unique fixed point.

**Proof:** Rearranging the given condition:  $d(F(x), F(y))(1 - (1 - \alpha)) \leq \alpha d(x, y)$ . Simplifying, we obtain:  $d(F(x), F(y)) \leq \frac{\alpha}{2-\alpha} d(x, y)$ . Since  $\frac{\alpha}{2-\alpha} < 1$ , this ensures contraction. Thus, by the Banach contraction principle,  $F$  has a unique fixed point.

### 3.3 Examples:

**Example 1:** Let  $X = [0,1]$  and define a fuzzy mapping  $F: X \rightarrow X$  by  $F(x) = \frac{x}{2}$ . The mapping satisfies the fuzzy contractive condition with  $\lambda = 0.5$ , ensuring the existence of a unique fixed point at  $x^* = 0$ .

**Example 2:** Consider  $X = \mathbb{R}$  with the rectangular metric  $d(x, y) = |x - y|^2$ . Define  $F(x) = \frac{x+1}{3}$ . This mapping is fuzzy contractive with  $\lambda = \frac{1}{3}$ , leading to a unique fixed point at  $x^* = \frac{1}{2}$ .

**Example 3:** Let  $X = \{0,1,2,3,4\}$  with a rectangular metric defined as  $d(x, y) = |x^2 - y^2|$ . Define  $F(x) = x - 1$  for  $x > 0$  and  $F(0) = 0$ . This function satisfies the contractive condition, and the fixed point is  $x^* = 0$ .

**3.4 Uniqueness of Fixed Points** If  $x^*, y^*$  are two fixed points, then  $d(x^*, y^*) = d(F(x^*), F(y^*)) \leq \lambda d(x^*, y^*)$ . Since  $\lambda < 1$ , this forces  $d(x^*, y^*) = 0$ , implying  $x^* = y^*$ .

## IV. APPLICATIONS

Fuzzy fixed point theorems in rectangular metric spaces have numerous applications in various scientific and engineering disciplines. Here, we explore a few significant applications along with their mathematical formulations:

**4.1 Optimization Problems** In optimization theory, finding fixed points corresponds to locating optimal solutions. Consider an iterative optimization method where updates are governed by a fuzzy contractive mapping:  $x_{n+1} = F(x_n)$  where  $F$  satisfies the contractive condition:  $d(F(x), F(y)) \leq \lambda d(x, y)$  for some  $\lambda \in [0,1)$ . Under suitable conditions, the iteration converges to a unique optimal solution.

**4.2 Image Processing** Fuzzy logic plays a crucial role in image enhancement and segmentation. Consider an image represented by a function  $I: \Omega \rightarrow [0,1]$ , where  $\Omega$  is the set of pixels. A smoothing transformation  $T$  based on fuzzy fixed points can be defined as  $T(I)(x) = \sum_{y \in \Omega} w(x, y)I(y)$  where  $w(x, y)$  is a fuzzy membership function satisfying:  $\sum_{y \in \Omega} w(x, y) = 1$ . Iterative applications of  $T$  converge to a stable image representation.

**4.3 Machine Learning and Neural Networks** Training of machine learning models involves minimizing an error function  $E(\theta)$  over parameters  $\theta$ . Gradient-based methods use update rules of the form:  $\theta_{n+1} = F(\theta_n) = \theta_n - \eta \nabla E(\theta_n)$  where  $\eta$  is the learning rate. If  $F$  satisfies a fuzzy contractive condition, convergence to an optimal set of parameters is guaranteed.

**4.4 Control Systems and Stability Analysis** In control theory, fuzzy fixed point results help analyze system stability. Consider a nonlinear control system governed by:  $x_{n+1} = f(x_n)$  where  $f$  represents the system dynamics. If  $f$  is fuzzy contractive, the system state converges to a stable equilibrium point, ensuring robustness and consistency in performance.

## V. CONCLUSION

This paper extends fixed-point theory to fuzzy mappings in rectangular metric spaces. We establish existence and uniqueness theorems under fuzzy contractive conditions. Future research can explore multi-valued fuzzy mappings and their applications in real-world systems.

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