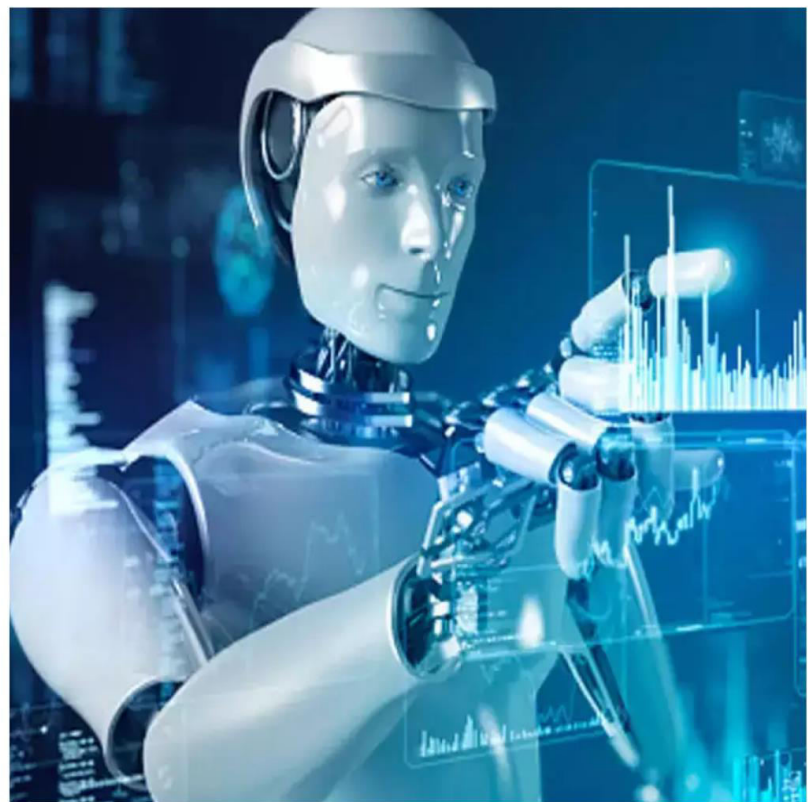




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Asymptotic Estimate of Level Crossing of Random Polynomials

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ABSTRACT: Let us consider a random Algebraic polynomial of the form

$$P(x) = P_n(x, w) = \sum_{i=0}^n a_i(w) \cos i\theta$$

in the interval $0 \leq \theta \leq 2\pi$. The expected number of level crossings of the random Polynomial of degree n with random real coefficients crosses the x -axis with independent normally distribution at K - times is any constant independent of x .

Several researchers have already done on the same polynomial that these asymptotic estimates cross the lines $P(x)=K$ in the interval (α, β) . Our predecessors such as [3],[4] have proved that the coefficients of a_i of $P(x)$ are independent standard normal random variables and the number of zeros within the interval $(-\infty, \infty)$ was $EN(-\infty, \infty) = \sqrt{n-1}$.

We consider the same algebraic polynomial whose contents are Binomial expression in the form $\binom{n-1}{r}$ and prove that the number of real roots of the equation $E(X)=K$ is asymptotically reducing to $EN(-\infty, \infty) = \sqrt{n-2} + 0(1)$ which was better approximation to the previous results.

KEYWORDS AND PHRASES: Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots.

I. INTRODUCTION

Let (Ω, A, Pr) be a fixed probability space and let $\{a_i(w)\}_{i=0}^{n-1}$ be a sequence of independent random variables defined on Ω . The random algebraic polynomial was introduced in the pioneer work of Littlehood and Offord [6] by considering the polynomial of the form

$$f(x) = f_n(x, w) = \sum_{i=0}^{n-1} a_i(w)x^i \quad \dots\dots\dots (1.1)$$

Our predecessors such as Bag [4], Kac [5] & Little[6] they consider the same polynomial with the coefficients $a_i(w)$ have variance $(1/i)!$ and they consider the line $P(x)=K$ and for n sufficiently larger and found that the expected number of level crossings of the polynomial (1.1) with the line $P(x)=K$ in the interval $(-\infty, \infty)$ is approximate to $0(\sqrt{n})$ i.e.

$$EN(-\infty, \infty) = 0\sqrt{n} \quad \dots (1.2)$$

However, many researchers have substantial progress towards number of zeros between same polynomial with the line $y=k$ and they get same result in same interval. The expected number of zeros in $(-\infty, \infty)$ is equal to $\sqrt{n-1} + 0(1)$ and they proved the following theorems.

Theorem-1. If the coefficients a_i of $P(x)$ are independent standard normal random variables, then

$$EN_0(-\infty, \infty) = \sqrt{n-1}.$$

Theorem-2. Denote

$$\gamma(n) = (2n-1)!! / (2n)!! \text{ and assume } K^2 \gamma(n) \rightarrow 0.$$

With the same assumptions as in Theorem-1 for the coefficients of $P(x)$, we have

$$EN_0(-\infty, \infty) \sim \sqrt{n-1}.$$

Theorem-3. Denote $M(-\infty, \infty)$ the number of extreme of $P(x)$ in $(-\infty, \infty)$. Then the expected value of $M(-\infty, \infty)$ satisfies

$$EN(-\infty, \infty) \sim \sqrt{n-1}.$$

Our work

We consider a polynomial of the form

$$Q(x) = Q_n(x, w) = \sum_{i=0}^{n-1} a_i(w) \binom{n-1}{i}^{1/2} x^i. \quad \dots\dots (1.3)$$

Where the coefficients a_i 's are binomial expression in the form $\sqrt{\binom{n-1}{i}}$ and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, and the polynomial in the form

$$Q(x) = a_0(w) \sqrt{\binom{n-1}{0}} + a_1(w) \sqrt{\binom{n-1}{1}} x + a_2(w) \sqrt{\binom{n-1}{2}} x^2 + \dots + a_{n-1}(w) \sqrt{\binom{n-1}{n-1}} x^{n-1} \quad \dots\dots (1.4)$$

and prove that the number of real roots of the equation $E(X)=K$ with the polynomial

$$Q(x) = Q_n(x, w) = \sum_{i=0}^{n-1} a_i(w) \binom{n-1}{i}^{1/2} x^i. \text{ is asymptotically reduces to } EN(-\infty, \infty) = \sqrt{n-2} + 0(1)$$

which was better approximation to the previous results.

II. A FORMULA FOR THE EXPECTED NUMBER OF REAL ROOTS

Let

$$A^2 = \text{Var} \{Q(x)-K\} \quad \dots\dots (2.1)$$

$$B^2 = \text{Var} \{Q'(x)\} \quad \dots\dots (2.2)$$

$$C = \text{Cov} [\{Q(x)-K\}, Q'(x)] \quad \dots\dots (2.3)$$

and

$$\eta = -CK / A\sqrt{A^2 B^2 - C^2}.$$

Then by using the expected number of level crossings given by Cramer and Lead better [2] for our equation $P(x)-K=0$, we can obtain

$$EN_k(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{B\sqrt{1-C^2/A^2B^2}}{A} \Phi\left(-\frac{K}{A}\right) [2\phi(\eta) + \eta\{2\phi(\eta) - 1\}] dx,$$

where as usual

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^t \exp(-y^2/2) dy$$

and

$$\Phi(t) = (2\pi)^{-1/2} \exp(-t^2/2).$$

Let $\Delta^2 = A^2B^2 - C^2$ and $\text{erf}(x) = \int_0^x \exp(-t^2) dt$; then we can write the extension of a formula obtained by Rice

[7] for the case of $K=0$ as

$$EN_k(\alpha, \beta) = I_1(\alpha, \beta) + I_2(\alpha, \beta), \tag{2.4}$$

where

$$I_1(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{\Delta}{\pi A^2} \exp\left(\frac{B^2K^2}{2\Delta^2}\right) dx \tag{2.5}$$

and

$$I_2(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{\sqrt{2KC}}{\pi A^2} \exp\left(-\frac{K^2}{2\Delta^2}\right) \text{erf}\left(\frac{KC}{\sqrt{2A\Delta}}\right) dx.$$

III. PROOF OF THEOREM-1

$$A^2 = \sum_{i=0}^{n-1} \binom{n-1}{i} x^{2i} = (x^2 + 1)^{n-1} \tag{3.1}$$

Since $\frac{d}{dt} (1+t)^{n-1} = \frac{d}{dt} \sum_{i=0}^{n-1} \binom{n-1}{i} t^i = \sum_{i=0}^{n-1} \binom{n-1}{i} i t^{i-1}$

$$\Rightarrow t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i \binom{n-1}{i} t^i$$

Similarly, $\frac{d^2}{dt^2} (1+t)^{n-1} = \sum_{i=0}^{n-1} i(i-1) \binom{n-1}{i} t^i$

$$\Rightarrow t^2(n-1)(n-2)(1+t)^{n-3} = \sum_{i=0}^{n-1} i(i-1) \binom{n-1}{i} t^i$$

$$\Rightarrow t^2(n-1)(n-2)(1+t)^{n-3} + t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i^2 \binom{n-1}{i} t^i$$

$$B^2 = \sum_{i=0}^{n-1} i^2 \binom{n-1}{i} x^{2i-2}$$

$$= x^{-2} \sum_{i=0}^{n-1} i^2 \binom{n-1}{i} x^{2i}$$

$$= x^{-2} (x^4(n-1)(n-2)(1+x^2)^{n-3} + x^2(n-1)(1+x^2)^{n-2})$$

$$= x^2(n-1)(n-2)(1+x^2)^{n-3} + (n-1)(1+x^2)^{n-2}$$

$$= (n-1)(x^2+1)^{n-3} (nx^2 - x^2 + 1) \tag{3.2}$$

And

$$C = \sum_{i=0}^{n-1} i \binom{n-1}{i} x^{2i-1}$$

$$\begin{aligned} &=x^{-1}\sum_{i=0}^{n-1}i\binom{n-1}{i}x^{2i} \\ &=x^{-1}(x^2(n-1)(1+x^2)^{n-2}) = (n-1)x(x^2+1)^{n-2} \quad \dots (3.3) \end{aligned}$$

We used the following, where (3.2) and (3.3) are obtained by differentiation of (3.1). Therefore, since from (3.1), (3.3)

$$\begin{aligned} \Delta^2 &= A^2B^2 - C^2 \\ &= (x^2+1)^{n-1}(n-1)(x^2+1)^{n-3}(nx^2-x^2+1) - \{(n-1)x(x^2+1)^{n-2}\}^2 \\ &= (x^2+1)^{2n-4}(n-1)((n-1)x^2+1) - (n-1)^2x^2(x^2+1)^{2n-4} \\ &= (n-1)^2x^2(x^2+1)^{2n-4} + (x^2+1)^{2n-4}(n-1) - (n-1)^2x^2(x^2+1)^{2n-4} \\ &= (n-1)(x^2+1)^{2n-4} \quad \dots\dots\dots (3.4) \end{aligned}$$

from (2.4) -(2.6) we obtain

$$EN_n(0, \infty) = \pi^{-1} \int_0^\infty \frac{\Delta}{A^2} dx = \frac{\sqrt{n-1}}{\pi} \int_0^\infty \frac{dx}{x^2+1} = \frac{\sqrt{n-1}}{2} \quad \dots (3.5)$$

This gives the proof of Theorem 1. It is indeed, interesting to note that from (3.5)

$$\frac{\sqrt{n-1}}{\pi(x^2+1)}$$

is the density of the number of real zeros of Q(x); see also Nayak & Pratihari [3]. Obtaining a closed form of the above density function is uncommon. An asymptotic result, for most cases, is the best that can be achieved.

IV. PROOF OF THE THEOREM-2

Because $|erf u| < \sqrt{\pi/2}$, it follows that from (2.6), (3.1) that

$$\begin{aligned} 0 \leq I_2(-\infty, \infty) &\leq \frac{|K|}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{C}{A^3} \exp\left(-\frac{K^2}{2A^2}\right) dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{|K|/\sqrt{2}} e^{-v^2} dv, \end{aligned}$$

if $v = |K|/(A\sqrt{2})$. Therefore, because $erf u \leq u$ when $u \geq 0$,

$$0 \leq I_2(-\infty, \infty) \leq \frac{2}{\sqrt{\pi}} erf\left(\frac{K}{\sqrt{2}}\right) \leq \min\left\{1, \frac{\sqrt{2}|K|}{\sqrt{\pi}}\right\} \quad \dots (4.1)$$

Moreover, it follows from (2.5), (3.1), (3.2) and (3.4) that

$$I_1(-\infty, \infty) = \frac{\sqrt{n-1}}{\pi} \int_{-\infty}^\infty \frac{e^{-s}}{x^2+1} dx,$$

in which $s = K^2(nx^2 - x^2 + 1)/\{2(x^2 + 1)^{n-1}\}$. If $x = \tan\theta$, we find that

$$I_1(-\infty, \infty) = \frac{2\sqrt{n-1}}{\pi} \int_{-\infty}^{\infty} e^{-s} d\theta,$$

in which $s = (K^2 / 2)\{(n-1)\sin^2 \theta + \cos^2 \theta\} \cos^{2n-4} \theta$. Therefore,

$$I_1(-\infty, \infty) = \frac{2\sqrt{n-1}}{\pi} \int_0^{\pi/2} \{1 - (1 - e^{-s})\} d\theta = \sqrt{n-1} - R,$$

$$0 \leq R = \frac{2\sqrt{n-1}}{\pi} \int_0^{\pi/2} (1 - e^{-s}) d\theta \leq 2\sqrt{n-1} \int_0^{\pi/2} s d\theta = K^2 \beta_n$$

in which

$$\beta_n = \frac{\sqrt{n-1}(3n-4)(2n-5)!!}{(2n-2)(2n-4)!!} = \sqrt{n-1} \gamma(n-2) \frac{3n-4}{2n-2}.$$

A straight forward algebraic calculation shows that $\beta_{n+1} < \beta_n$ when $n \geq 2$. We conclude that $0 \leq R \leq K^2 \beta_2 = K^2/2$ and then that $R = 0(\sqrt{n})$ because $K^2 = O\{\gamma^{-1}(n)\}$ and $\gamma(n) \sim \sqrt{n\pi}$. When this last result is combined with (4.1), (4.2) and (2.4), it is clear that theorem 2 is true. In fact, we have actually proved the better result that

$$\frac{K^2}{2} \leq EN_k(-\infty, \infty) - \sqrt{n-1} \leq \min \left\{ 1, \frac{\sqrt{2|K|}}{\sqrt{\pi}} \right\},$$

from which we can also infer not only Theorem 2 but also that

$$EN_k(-\infty, \infty) = \sqrt{n-1} + O(1)$$

when K is bounded.

V. EXTREMA

The expected number of extrema of $Q(x)$, denoted by $EN_k(-\infty, \infty)$ is simply the expected number of real zeros of $Q(x) = \sum_{i=1}^{n-1} a_i i \binom{n-1}{i}^{1/2} x^{i-1}$. Therefore, we apply $EN_k(-\infty, \infty)$ for $Q'(x)$. To this end, by successive differentiation of (2.2)

We obtain

$$A^2 = \sum_{i=0}^{n-1} \binom{n-1}{i} x^{2i} = (x^2 + 1)^{n-1}$$

$$\begin{aligned} \text{Since } \frac{d}{dt} (1+t)^{n-1} &= \frac{d}{dt} \sum_{i=0}^{n-1} \binom{n-1}{i} t^i = \sum_{i=0}^{n-1} \binom{n-1}{i} i t^{i-1} \\ &\Rightarrow t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i \binom{n-1}{i} t^i \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \frac{d^2}{dt^2} (1+t)^{n-1} &= \sum_{i=0}^{n-1} i(i-1) \binom{n-1}{i} t^i \\ &\Rightarrow t^2(n-1)(n-2)(1+t)^{n-3} = \sum_{i=0}^{n-1} i(i-1) \binom{n-1}{i} t^i \end{aligned}$$

$$\Rightarrow t^2(n-1)(n-2)(1+t)^{n-3} + t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i^2 \binom{n-1}{i} t^i \quad \dots (5.1)$$

$$B^2 = \sum_{i=0}^{n-1} i^2 \binom{n-1}{i} x^{2i-2}$$

$$= (n-1)(x^2+1)^{n-3}(nx^2-x^2+1)$$

Since $\frac{d^2}{dt^2}(1+t)^{n-1} = \sum_{i=0}^{n-1} i(i-1)\binom{n-1}{i} t^{i-3}$

$$\Rightarrow t^3(n-1)(n-2)(n-3)(1+t)^{n-4} = \sum_{i=0}^{n-1} (i^3-3i^2+2i)\binom{n-1}{i} t^i$$

$$\Rightarrow t^3(n-1)(n-2)(n-3)(1+t)^{n-4} + 3t^2(n-1)(n-2)(1+t)^{n-3} + t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i^3 \binom{n-1}{i} t^i$$

.....(5.2)

$$C = \sum_{i=0}^{n-1} i^2(i-1)\binom{n-1}{i} x^{2i-2}$$

$$= x^{-2} \left(\sum_{i=0}^{n-1} i^3 \binom{n-1}{i} x^{2i} - \sum_{i=0}^{n-1} i^2 \binom{n-1}{i} x^{2i} \right)$$

$$= x^{-2} (x^6(n-1)(n-2)(n-3)(1+x^2)^{n-4} + x^4(n-1)(n-2)(1+x^2)^{n-3})$$

$$= (x^4(n-1)(n-2)(n-3)(1+x^2)^{n-4} + 2x^2(n-1)(n-2)(1+x^2)^{n-3})$$

$$= (n-1)(n-2)(x^2+1)^{n-5} \{ (n-1)(n-2)x^4 + 4(n-2)x^2 + 2 \}$$

... (5.3)

Therefore, from (3.4) and (5.1) -(5.3) we obtain

$$\frac{\Delta}{A^2} = \frac{\sqrt{(n-2)\{(n-1)(n-2)x^4 + 2(n-1)x^2 + 2\}}}{(x^2+1)\{(n-1)x^2 + 1\}} \quad \dots (5.4)$$

From (5.4) it then follows that, for all non-zero x

$$\lim_{n \rightarrow \infty} \frac{\Delta}{A^2 \sqrt{n-2}} dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} \quad \dots (5.5)$$

Then since $\Delta / A^2 \sqrt{n-2} \leq \sqrt{2} / (x^2+1)$ for all real x, the dominated convergent theorem for Lebesgue integrals shows that

$$\int_{-\infty}^{\infty} \frac{\Delta}{A^2 \sqrt{n-2}} dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi. \quad \dots (5.6)$$

Theorem 3 is then an immediate consequence of this result.

VI. CONCLUSION

We consider a different polynomial of the form

$$Q(x) = Q_n(x, w) = \sum_{i=0}^{n-1} a_i(w) \binom{n-1}{i}^{1/2} x^i. \quad \text{Where the}$$

coefficients a_i 's are binomial expression in the form $\sqrt{\binom{n-1}{i}}$ and prove that the number of real roots of the equation

$E(X)=K$ with the polynomial $Q(x)$ is asymptotically reduces

to $EN(-\infty, \infty) = \sqrt{n-2} + O(1)$.

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