SUBORDINATION RESULTS FOR CERTAIN SUBCLASSES OF UNIVALENT MEROMORPHIC FUNCTIONS

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Abstract: In this paper we consider \((1-\lambda)zg(z) \) of meromorphic functions \(g\) such that \(\text{Re}(\lambda g(z)) > \alpha\) be the class of \(\lambda -\) starlike functions of order \(\alpha\) (\(0 \leq \alpha < 1\)), and analogous to the class \(S^*_\alpha(\phi)\), for \(g \in \Sigma\), we define the class \(F^*[\beta, \lambda](\phi)\) of meromorphic functions \(g \in \Sigma\) then, obtaining some subordination results on functions in this class.

Keywords: Analytic functions, univalent functions, Starlike functions of complex order, Meromorphic functions.

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I. INTRODUCTION

Let \(A\) be the class of functions \(f\) normalized by

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

(1.1)

which are analytic in the open unit disc \(U = \{z \in \mathbb{C}: |z| < 1\}\) and let \(S\) be the subclass of \(A\) consisting of functions which are also univalent in \(U\).

Let \(\phi(z)\) be an analytic function with positive real part on \(U\) that satisfies \(\phi(0)=1\), \(\phi'(0) > 0\) and which maps the unit disc \(U\) onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Ma and Minda [2] introduced and studied the class \(S^*(\phi)\) consists of functions \(f \in S\) for which

\[
\frac{|zf'(z)|}{f(z)} < \phi(z) \quad (z \in U).
\]

Following Ma and Minda [2], Ravichandran et. al [9] defined a more general class \(S^*_\beta(\phi)\) of starlike functions of complex order consists of functions \(f \in S\) for which
\[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) < \phi(z) \quad (z \in U), \]

where \( b \neq 0 \) is a complex number.

Let \( \Sigma \) be the class of meromorphic functions \( g \) normalized by

\[ g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.2} \]

which are analytic and univalent in the punctured open unit disc \( U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = U - \{0\} \), where \( U = \{ z \in \mathbb{C} : |z| < 1 \} \).

The function \( g \in \Sigma \) is said to be meromorphic starlike of order \( \alpha \) \((0 \leq \alpha < 1)\), if \( g(z) \neq 0 \) and

\[ -\text{Re} \left( \frac{zg''(z)}{g'(z)} \right) > \alpha \quad (z \in U^*), \]

and this class is denoted by \( MS^*(\alpha) \) or \( \Sigma^*(\alpha) \). Where first introduced by Pommerenke [7], denoting it by \( \Sigma(\alpha) \).

Similarly the function \( g \in \Sigma \) is said to be meromorphic convex of order \( \alpha \) \((0 \leq \alpha < 1)\), if \( g'(z) \neq 0 \) and

\[ -\text{Re} \left( 1 + \frac{zg'(z)}{g(z)} \right) > \alpha \quad (z \in U^*), \]

and this class is denoted by \( MC(\alpha) \). Where also, first introduced by Pommerenke [7], denoting it by \( K(\alpha) \).

In this paper for functions \( g \in \Sigma, g(z) \neq 0 \) given as in (1.2) we consider \( -\text{Re} \left( \frac{zg''(z)}{g'(z)} \right) > \alpha \) be the class of meromorphic \( \lambda \)-starlike functions of order \( \alpha \) \((0 \leq \alpha < 1)\). Taking \( \lambda = 0 \) it reduce to the class \( \frac{zg''(z)}{g'(z)} \) with

\[ -\text{Re} \left( \frac{zg''(z)}{g'(z)} \right) > \alpha \quad (0 \leq \alpha < 1) \]

of meromorphic starlike functions of order \( \alpha \), and then analogous to the class \( S^\alpha_\beta(\phi) \), for \( g \in \Sigma \) we define the class \( F^*[b, \lambda](\phi) \) of meromorphic functions \( g \in \Sigma \) as follows

**Definition 1.1**

Let \( \phi(z) \) be an analytic function with positive real part on \( U \) that satisfies \( \phi(0) = 1, \phi'(0) > 0 \) and which maps the unit disc \( U \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let \( F^*[b, \lambda](\phi) \) be the class of functions \( g, g \in \Sigma, g(z) \neq 0 \) satisfying
where 0 ≤ λ < 1 and b ≠ 0 is a complex number.

Taking λ = 0 our class \( F^*[b, \lambda](\phi) \) reduce to the class \( M^*_b(\phi) \) defined by Mohammed and Darus [6].

Using mainly the method of subordination we obtain some results for the class \( F^*[b, \lambda](\phi) \).

Let \( F \) and \( G \) be analytic functions in the unit disc \( U \). The function \( F \) is subordinate to \( G \) written \( F \prec G \) if \( G \) is univalent, \( F(0) = G(0) \) and \( F(U) \subset G(U) \). In general, given two functions \( F \) and \( G \) which are analytic in \( U \), the function \( F \) is said to be subordinate to \( G \) if there exist a function \( w \) analytic in \( U \) with \( w(0)=0 \) and \(|w(z)| < 1\), such that \( F(z) = G(w(z)) \).

The general theory of differential subordinations was introduced by Miller and Mocanu [3] (see also [4] and [5]).

Our results and their proofs are motivated by similar manners as results of Srivastava and Lashin [10], Ravichandran et al [9], Ramachandran [8], Ali and Ravichandran [1] and Mohammed and Darus [6].

To prove our main results, the following lemmas are needed

**Lemma 1.2** [5] Let \( \phi \) be a convex function defined on \( U \), \( \phi(0)=1 \). Define \( F(z) \) by

\[
F(z) = z \exp \left( \int_0^z \frac{\phi(x)}{x} \, dx \right).
\]

Let \( q(z) = 1 + cz^2 + cz^3 + \ldots \) be analytic in \( U \). Then

\[
1 + \frac{zq'(z)}{q(z)} \prec \phi(z),
\]

if and only if for all \(|s| < 1\) and \(|t| < 1\), we have

\[
\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}.
\]

**Lemma 1.3** [4] Let \( q(z) \) be univalent in the unit disc \( U \) and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = zq'(z) \varphi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \).

Suppose that either \( h(z) \) is convex, or \( Q(z) \) is starlike univalent in \( U \). In addition, assume that \( \Re \left( \frac{zQ'(z)}{Q(z)} \right) > 0 \) for \( z \in U \). If \( p(z) \) is analytic in \( U \) with \( p(0) = q(0) \), \( p(U) \subset D \) and

\[
\frac{p(tz)}{p(sz)} \prec \frac{sF(tz)}{tF(sz)}.
\]
\[ \partial (p(z)) + zp'(z) < \partial (q(z)) + zq'(z), \]
then \( p(z) < q(z) \) and \( q(z) \) is the best domenant.

II. SUBORDINATION RESULTS

First we prove the following result for functions in the class \( F^*[b,\lambda](\phi) \).

Theorem 2.1

A function \( g \) for \( g(z) \neq 0 \) if and only if

\[ [z^\frac{1}{1+\lambda} g(z)]^\frac{1}{\lambda} = \exp \left( \int_0^z \frac{1-\phi(w(\xi))}{\xi} \, d\xi \right), \]

where \( w(z) \) is analytic in \( U \) satisfying \( w(0)=0 \) and \( \text{Im}(w(z)) \leq 1 \).

Proof:

First let \( g \in F^*[b,\lambda](\phi) \), applying Definition 1.1. We get

\[ 1 - \frac{1}{b} \left( \frac{(1-\lambda) g'(z)}{\lambda g(z) + g(z)} + 1 \right) < \phi(z), \quad (2.2) \]
therefore, there is a function \( w(z) \) analytic in \( U \) with \( w(0)=0 \) and \( \text{Im}(w(z)) \leq 1 \) such that

\[ 1 - \frac{1}{b} \left( \frac{(1-\lambda) g'(z)}{\lambda g(z) + g(z)} + 1 \right) = \phi(w(z)), \quad (2.3) \]
or

\[ \frac{1}{b} \left( \frac{(1-\lambda) g'(z)}{\lambda g(z) + g(z)} + 1 \right) = \frac{1-\phi(w(z))}{z}. \quad (2.4) \]

Integrating both sides of (2.4) from 0 to \( z \) gives

\[ \frac{1}{b} \left( \frac{(1-\lambda) \ln g(z) + \ln z}{\lambda + 1} \right) = \int_0^z \frac{1-\phi(w(\zeta))}{\zeta} \, d\zeta, \quad (2.5) \]
so

\[ \frac{1}{b} \ln [z^\frac{1}{1+\lambda} g(z)]^\frac{1}{\lambda} = \int_0^z \frac{1-\phi(w(\zeta))}{\zeta} \, d\zeta, \quad (2.6) \]

Hence the result

\[ [z^\frac{1}{1+\lambda} g(z)]^\frac{1}{\lambda} = \exp \left( \int_0^z \frac{1-\phi(w(\zeta))}{\zeta} \, d\zeta \right). \quad (2.7) \]

Conversely, suppose that
Now differentiating (2.8) with respect to \( z \) we obtain

\[
\frac{1}{b} \left[ z(g(z))^{\frac{1-\lambda}{1+\lambda}} \right] = \exp \left( \int_0^x \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right)
\]  

(2.9)

begin{split}
\text{from (2.8) and (2.9) we obtain}
\frac{1}{b} \left( \frac{(1-\lambda)g'(z)}{1+\lambda g(z)} + 1 \right) = \frac{1-\phi(w(z))}{z}.
\end{split}

(2.10)

Hence

\[
1 - \frac{1}{b} \left( \frac{(1-\lambda)g'(z)}{\lambda g(z) + g(z)} + 1 \right) \leq \phi(z).
\]

(2.11)

Therefore

\[
g \in F'[b, \lambda](\phi).
\]

(2.12)

This completes the proof of Theorem 2.1.

Taking \( \lambda = 0 \) in Theorem 2.1 gives Theorem 2.1 in [6], then we have the following corollary.

**Corollary 2.2**

A function \( g \in F'[b, \lambda](\phi) \) for \( g(z) \neq 0 \) if and only if

\[
(zg(z))^{\frac{1}{(1-\lambda)}} = \exp \left( \int_0^x \frac{1-\phi(w(\zeta))}{\zeta} d\zeta \right)
\]

(2.13)

where \( w(z) \) is analytic in \( U \) satisfying \( w(0)=0 \) and \( |w(z)| \leq 1 \).

Using Lemma 1.2 we obtain the following necessary and sufficient conditions for functions to belong to \( F'[b, \lambda](\phi) \).

**Theorem 2.3**

Let \( \phi(\zeta) \) and \( F(z) \) be as in Lemma 1.2. The function \( g \in F'[b, \lambda](\phi) \) if and only if for all

\[
|s| \leq 1 \text{ and } |t| \leq 1,
\]

\[
\frac{1}{b} \left[ \frac{g(z)}{(1-\lambda)^{1+\lambda}} \right] = \frac{dF(z)}{F(z)}
\]

(2.14)
Proof:

Let $g \in F^*[b, \lambda](\varphi)$

Define the function $p(z) \neq 0$ by

$$\frac{1}{p(z)} = \frac{1}{[z(g(z))^{1+\lambda}]^{\lambda}}.$$  \hspace{1cm} (2.15)

Differentiating (2.15) with respect to $z$ we get

$$-\frac{p'(z)}{p(z)^2} = \frac{1}{b} \left[ z(g(z))^{1+\lambda} \right]^{b-1} \left[ (g(z))^{1+\lambda} \frac{1-\lambda z g'(z)}{1+\lambda g(z)} + 1 \right].$$  \hspace{1cm} (2.16)

Dividing (2.16) by (2.15) gives

$$\frac{-p'(z)}{p(z)} = \frac{1}{b} \left[ g(z) \right]^{1+\lambda} \frac{1-\lambda z g'(z)}{1+\lambda g(z)} + 1,$$

or

$$1 + \frac{z p'(z)}{p(z)} = 1 - \frac{1}{b} \left[ 1 - \frac{1-\lambda z g'(z)}{1+\lambda g(z)} + 1 \right].$$  \hspace{1cm} (2.17)

then by Definition 1.1

$$1 + \frac{z p'(z)}{p(z)} = 1 - \frac{1}{b} \left[ 1 - \frac{1-\lambda z g'(z)}{1+\lambda g(z)} + 1 \right] < \varphi(z).$$  \hspace{1cm} (2.18)

And applying Lemma 1.2, we get that

$$1 + \frac{z p'(z)}{p(z)} = 1 - \frac{1}{b} \left[ 1 - \frac{1-\lambda z g'(z)}{1+\lambda g(z)} + 1 \right] < \varphi(z),$$  \hspace{1cm} (2.19)

if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{p(tz)}{p(sz)} = \frac{1}{t^\lambda \left[ g(sz) \right]^{1+\lambda}} \frac{1}{s^\lambda \left[ g(tz) \right]^{1+\lambda}} < \frac{p(tz)}{t F(sz)}.$$  \hspace{1cm} (2.20)

This completes the proof of Theorem 2.3

If we take $\lambda = 0$ in Theorem 2.3 then we have the following corollary due to [6].

**Corollary 2.4**

Let $\varphi(z)$ and $F(z)$ be as in Lemma 1.2. The function $g \in F^*[b, \lambda](\varphi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,
Theorem 2.5

Let \( q(z) \) be univalent and \( q(z) \neq 0 \) in \( U \) and such that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \), and

\[
\text{Re}[1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}] > 0 \quad (z \in U, \gamma \neq 0).
\]  (2.23)

If \( g(z) \in \Sigma \) for \( g(z) \neq 0 \) and

\[
\frac{((1-\lambda)zg''(z))}{g(z)} - \gamma (1 + \frac{zg''(z)}{g(z)} - \frac{zg'(z)}{g(z)}) < q(z) - \gamma \frac{zq(z)}{q(z)}
\]  (2.24)

Then

\[
\frac{((1-\lambda)zg''(z))}{g(z)} < q(z).
\]  (2.25)

Proof:

Define the function \( p(z) \) by

\[
p(z) = \frac{(1-\lambda)zg'(z)}{g(z)+g(z)} \quad (z \in U),
\]  (2.26)

differentiating (2.26) with respect to \( z \), we obtain

\[
p'(z) = \frac{1-\lambda}{1+\lambda} \frac{g(z)[zg''(z)+g'(z)]-zg'(z)^2}{(g(z))^2},
\]  (2.27)

and

\[
\frac{zp'(z)}{p(z)} = \frac{zg(z)[zg''(z)+g'(z)]-z^2(g'(z))^2}{g(z)} \frac{g(z)}{zg'(z)}
\]  

\[
= \frac{zg''(z)}{g'(z)} + 1 - \frac{zg'(z)}{g(z)}
\]  (2.28)

so

\[
p(z) - \gamma \frac{zp'(z)}{p(z)} = \frac{1-\lambda}{1+\lambda} \frac{zg'(z)}{g(z)} - \gamma \frac{zg'(z)}{g'(z)} + \gamma \frac{zg'(z)}{g(z)} - \gamma
\]  

\[
= \frac{(1-\lambda)zg'(z)}{g(z) + g(z)} - \gamma \left( \frac{zg''(z)}{g'(z)} + 1 \right) + \gamma \frac{zg'(z)}{g(z)}.
\]  (2.29)
By setting 
\( \theta(w) = w \)

and

\( \phi(w) = -\frac{v}{w} \),

it can be easily observed that \( \theta(w) \) is analytic in the complex plane \( \mathbb{C} \), \( \phi(w) \) is analytic in the complex plane \( \mathbb{C} \setminus \{0\} \) and that

\[ \phi(w) \neq 0 \ (w \in \mathbb{C} \setminus \{0\}) \]

Also, by letting

\[ Q(z) = zq'(z)\phi(q(z)) = -\gamma \frac{zq'(z)}{q(z)} \]

and

\[ h(z) = \theta(q(z)) + Q(z) = q(z) - \gamma \frac{zq'(z)}{q(z)} \]

Thus

\[ \frac{zQ(z)}{Q(z)} = \frac{zq'(z)}{q(z)} + 1 - \frac{zq'(z)}{q(z)} \] \quad (2.30)

and

\[ \frac{zq'(z)}{q(z)} = \frac{q(z)[zq''(z)+q'(z)]-zq'(z)^2}{q(z)^2} \frac{q(z)}{q'(z)} \]

\[ = \frac{zq''(z)+1 - \frac{z}{q(z)}}{q(z)} \] \quad (2.31)

Since \( \frac{zq'(z)}{q(z)} \) is starlike and univalent in \( U \) so

\[ \text{Re}\left\{ \frac{zq''(z)+1 - \frac{z}{q(z)}}{q(z)} \right\} > 0 \] \quad (2.32)

Then, \( Q(z) \) is starlike and univalent in \( U \).

Now
And from (2.23) we know that
\[ \Re[1 + \frac{z q''(z)}{q(z)} - \frac{z q'(z)}{q(z)} - \frac{q(z)}{y}] > 0 \]

therefore, by an application of lemma 1.3 we get that
\[ p(z) = \frac{(1 - \lambda)z q'(z)}{\lambda g(z) + g(z)} < q(z), \quad (2.34) \]

and \( q(z) \) is the best dominant.

This completes the proof of Theorem 2.5.

If we take \( \lambda = 0 \), then we have the following corollary.

**Corollary 2.6**

Let \( q(z) \) be univalent and \( q(z) \neq 0 \) in \( U \) and such that \( \frac{z q'(z)}{q(z)} \) is starlike univalent in \( U \),

and
\[ \Re[1 + \frac{z g''(z)}{g(z)} - \frac{z g'(z)}{g(z)} - \frac{g(z)}{y}] > 0 \quad (z \in U, y \neq 0). \quad (2.35) \]

If \( g(z) \in \Sigma \) for \( g(z) \neq 0 \) and
\[ \frac{z g'(z)}{g(z)} - y \left(1 + \frac{z g'(z)}{g(z)} - \frac{z g'(z)}{g(z)} \right) < q(z) - \frac{z g'(z)}{q(z)}, \quad (2.36) \]

then
\[ \frac{z g'(z)}{g(z)} < q(z) \quad (2.37) \]

and \( q(z) \) is the best dominant.
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